# A Nonlocal Neumann Problem for Semilinear Elliptic Equations

NG, Chit Yu

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Thesis/Assessment Committee

Professor Kwok-Keung Au (Chair)

Professor Kai-Seng Chou (Thesis Supervisor)

Professor Yau-Heng Wan (Committee Member)

# Abstract


In this thesis, we study nonconstant solutions to the following problem

$$\begin{cases} -\Delta u = g(x,u) - \fint_{\Omega} g(x,u)\, dx & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g(x,u)$ is a perturbation of order $p$ at infinity uniform in $x$ and $p \in (1, (n+2)/(n-2)]$. Under various assumptions on the function $g$, existence, uniqueness and multiplicity results are established. We approach the problem by variational methods.

# 摘要


本文探討以下問題的非常數解：

$$\begin{cases} -\triangle u = g(x,u) - \fint_{\Omega} g(x,u)\, dx & \text{在 } \Omega \text{ 內,} \\ \dfrac{\partial u}{\partial n} = 0 & \text{在 } \partial\Omega \text{ 上,} \end{cases}$$

其中 $g(x,u)$ 是一個位置無限遠處的 $p$ 階擾動，且 $p$ 在區間 $(1,(n+2)/(n-2)]$ 內。對於不同條件下的 $g$ ，我們將以變分法推出對應解的存在性、唯一性或多重性。

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# Chapter 1

# Introduction

In this thesis we study the following nonlocal Neumann problem

$$\begin{cases} -\Delta u = g(x,u) - \fint_{\Omega} g(x,u)\, dx & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \fint_{\Omega} u \, dx = A, & \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n (n \geq 3)$ with $C^2$-boundary, $A$ is a real constant and $g$ is a real-valued function in $C^1(\overline{\Omega} \times \mathbb{R})$. Define $G(x,z) := \int_0^z g(x,t)\, dt$ for $x \in \overline{\Omega}$ and $z \in \mathbb{R}$. We study existence, uniqueness and multiplicity of solutions of Problem (1.1) for different $g$ and $G$ by exploring the fact that it is a critical point of the functional

$$\frac{1}{2}\int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} (G(x, u+A) - G(x,A))\, dx.$$

We are mostly interested in Problem (1.1) when $g(x,z) = |z|^{p-1}z + f(x,z)$, where $f(x,z)$ is a lower order perturbation of $|z|^p$ uniform in $x$. In this case, the problem can be rewritten as

$$\begin{cases} -\Delta u = |u|^{p-1}u + f(x,u) - \fint_\Omega |u|^{p-1}u + f(x,u)\,dx & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \fint_\Omega u\,dx = A, \end{cases} \tag{1.2}$$

where $0 < p \leq (n+2)/(n-2) = 2^* - 1$ and $2^*$ is the critical exponent for the Sobolev embedding theorem.

## 1.1 Background

In this thesis we assume basic knowledge in functional analysis and Sobolev space, and the readers may consult [4] and [5], [8] and [9] for background materials.

Our motivation for the study of (1.1) is based on the Cahn-Hilliard type equation

$$\begin{cases} u_t + \Delta\left(\Delta u + g(x,u)\right) = 0 & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial(\Delta u)}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{1.3}$$

where $g(x,z) := z^3 - z$ and $u(x,0)$ is given. The equation is called the Cahn-Hilliard equation and is a model on phrase separation in cooling binary alloys [3], [7]. [11]. Its "energy" is given by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega G(x,u)\,dx.$$

We have the energy dissipation relation

$$E(u(\cdot,t)) + \int_0^t \int_\Omega |\nabla\left(\Delta u(x,s) + g(x,u(x,s))\right)|^2\,dx\,ds = E(u(\cdot,0)).$$

The relation formally shows that if the solution is global and converges eventually to some steady state $w$, then $w$ satisfies

$$\int_\Omega |\nabla\left(\Delta w + g(x,w)\right)|^2\,dx = 0.$$

In other words, $w$ satisfies $\Delta w + g(x,w) =$ constant and is a solution to (1.1). Note that, we have

$$\frac{\partial}{\partial t}\int_\Omega u(x,t)\ dx = -\int_\Omega \Delta\left(\Delta u + g(x,u)\right)\ dx = 0, \tag{1.4}$$

which implies that initial mass $u_0\ dx$ is preserved by the equation.

The classification of the steady states of the one-dimensional Cahn-Hilliard equation is done in [11].

A second order version of (1.3) is the nonlocal problem

$$\begin{cases} u_t = \Delta u + g(x,u) - \displaystyle⨍_\Omega g(x,u)\ dx & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases}$$

where $u(x,0)$ is given, see [6]. Similar to (1.3), (1.4) also holds and the dissipation relation is replaced by

$$E(u(\cdot,t)) + \int_0^t\int_\Omega \left(\Delta u(x,s) + g(x,u(x,s)) - ⨍_\Omega g\right)^2\ dx\ ds = E(u(\cdot,0)).$$

Again, its steady states are also solutions to (1.1).

## 1.2 Variational formulation

We are going to tackle the problem via variational methods. We have to choose a suitable space on which we can do calculus of variations. The set

$$\left\{u\in H^1(\Omega) : ⨍_\Omega u\ dx = A\right\}$$

is not a Hilbert space but an affine subset of $H^1(\Omega)$, unless $A=0$. We consider the following equivalent problem

$$\begin{cases} -\Delta u = g(x,u+A) - \displaystyle⨍_\Omega g(x,u+A)\ dx & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \displaystyle⨍_\Omega u\ dx = 0, \end{cases} \tag{1.5}$$

where $\Omega$, $g$ and $A$ are as before. We can associate problem (1.5) with a variational principle on the space

$$X := \left\{ u \in H^1(\Omega) : \fint_\Omega u \, dx = 0 \right\}, \tag{1.6}$$

which, by Poincare inequality, is a Hilbert space with the inner product

$$\langle u, v \rangle = \int_\Omega \nabla u \cdot \nabla v \, dx.$$

The variational principle is given by the critical point of the following energy functional $J$ on $X$.

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega (G(x, u + A) - G(x, A)) \, dx. \tag{1.7}$$

We say that $u \in X$ is a critical point of the energy $J$ if $J'(u)\varphi = 0$ for all $\varphi \in X$. Note that

$$J'(u)\varphi = \int_\Omega \nabla u \cdot \nabla \varphi \, dx - \int_\Omega g(x, u + A)\varphi \, dx. \tag{1.8}$$

## 1.3 Organization of the thesis

This thesis is organized as follows. In Chapter 2, we are concerned with the regularity of critical points to the energy $J$ given by (1.7) in different situations.

In Chapter 3, we deal with the sublinear case of the problem (1.1), that is, when $g(x, z) \in C^1(\overline{\Omega} \times \mathbb{R})$ satisfies the growth control $|g(x, z)| \leq C(1 + |z|^\gamma)$ on $\overline{\Omega} \times \mathbb{R}$ for some exponent $\gamma \in (0, 1)$. We will show the existence of a solution via minimization.

In Chapter 4, we show the existence of solution in the superlinear-subcritical case of the problem (1.1) with the particular form of nonlinearity $g(x, z) = |z|^{p-1}z + f(x)$ for some nonconstant $f \in C^1(\overline{\Omega})$, where $2 < p + 1 < \min\{3, 2^*\}$.

We will find solutions using the method of Nehari manifold.

Observe that when $f(x)$ above is a constant function, the problem (1.1) always has the constant $A$ as a trivial solution. More generally, when $g(x, A) \equiv$ constant, the problem (1.1) also has the constant $A$ as the trivial solution. In this case, we are interested in the existence of nontrivial solutions. In Chapter 5, we derive the existence of nontrivial solution via the mountain pass lemma under certain subcritical growth conditions on the nonlinearity $g$ as well as the smallness of $A$

In Chapter 6, we prove further multiplicity results for the subcritical case of (1.1) by the notion of genus when $A = 0$ and the nonlinearity $g(x, z)$ is odd in $z$. In this case, $J$ defined in (1.7) is even and hence there is a $\mathbb{Z}_2$ symmetry on $X$.

Finally, in Chapter 7, we prove the existence of nontrivial solution for the critical case with a special form of nonlinearity. More precisely, when $g(x, z) = |z|^{p-1}z + \lambda z + \mu|z|^{q-1}z$ for some $q \in (1, p) = ((1, (n+2)/(n-2))$ and $\lambda$, $\mu \geq 0$. The arguments in Chapter 5 do not work here due to the lack of compactness. The approach and techniques employed in this chapter are strongly motivated by [2] and [13]. The arguments involve a huge amount of computations and we place them in Appendix C in order to keep the presentation smooth.

# Chapter 2

# Regularity of The Critical Point

As we have mentioned in section 1.2, we transform the problem of finding solutions to the problem of searching critical points to an associated energy functional. But before looking for critical points to the energy functional, we have to first make sure that whether a critical point is a classical solution or not. In this chapter, we are going to solve this fundamental problem.

## 2.1 Regularity for $1 < p \leq (n+2)/(n-2)$

To establish the regularity result, we need two regularity lemmas. In the following lemma, the first one is a modified version of an analogous result in [13] and the second one is taken from [9].

**Lemma 2.1.1.** *Let $\Omega$ be an open, bounded and connected subset of $\mathbb{R}^n$ $(n \geq 3)$.*

*(i) Let $u$ be a weak solution of*

$$\begin{cases} -\Delta u = a(x)u + d & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

*where $\partial\Omega$ is $C^1$, $a(x) \in L^{n/2}(\Omega)$ and $d$ is a constant. Then $u \in L^q(\Omega)$ for all $q \geq 1$.*

*(ii) If $u$ is a weak solution of*

$$\begin{cases} -\Delta u = h(x) + d & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

*where $\partial\Omega$ is $C^2$, $h(x) \in L^p(\Omega)$, $p \in (1, \infty)$ and $d$ is a constant. Then $\|u\|_{2,p,\Omega} \le C\left(\|h(x) + d\|_{p,\Omega} + \|u\|_{1,\Omega}\right)$.*

**Remark 2.1.2.** (i) By a weak solution of (2.1), we mean a function $u$ in $H^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega a(x) u \varphi \, dx + d \int_\Omega \varphi \, dx, \tag{2.3}$$

for all $\varphi \in H^1(\Omega)$.

(ii) By a weak solution of (2.2), we mean a function $u$ in $H^1(\Omega)$ satisfying (2.3) for $a(x)u + d$ replaced by $h(x) + d$.

*Proof.* We first fix any $x_0 \in \overline{\Omega}$ and $\delta > 0$. Let $\eta$ be a nonnegative smooth bump centered at $x_0$ with support $B(x_0, \delta)$ and $\eta \equiv 1$ on $B(x_0, \delta/2)$.

We further fix $\beta > 1$, $N > 0$ and set $G \in C^1(\mathbb{R})$ by $G(t) = |t|^\beta$ for $|t| \le N$ and $G(t)$ is linear for $|t| > N$. Suppose $u$ is a weak solution of (2.1). Then $G(u)$, $G'(u)$, $F(u) = \int_o^u |G'(t)|^2 \, dt$ all belong to $H^1(\Omega)$. We may put $\varphi = F(u)\eta^2$ into (2.3) to get

$$\begin{aligned} \int_\Omega |G'(u)|^2 \eta^2 |\nabla u|^2 \, dx &+ \int_\Omega 2F(u)\eta \nabla u \cdot \nabla \eta \, dx \\ &= \int_\Omega a(x) u F(u) \eta^2 \, dx + d \int_\Omega F(u) \eta^2 \, dx. \end{aligned} \tag{2.4}$$

Note that $F(u) = \int_0^u G'(t) dG(t) = G'(u)G(u) - \int_0^u G'(t) dG(t)$ and hence $F(u) = \frac{1}{2} G'(u) G(u)$. This implies

$$\begin{aligned} \left| \int_\Omega 2F(u)\eta \nabla u \cdot \nabla \eta \, dx \right| &= \left| \int_\Omega G'(u) G(u) \eta \nabla u \cdot \nabla \eta \, dx \right| \\ &\le \varepsilon \int_\Omega |G'(u)|^2 \eta^2 |\nabla u|^2 \, dx + \frac{1}{4\varepsilon} \int_\Omega |G(u)|^2 |\nabla \eta|^2 \, dx. \end{aligned} \tag{2.5}$$

Putting it into (2.4), we obtain

$$
\begin{aligned}
&(1-\varepsilon)\int_\Omega |G'(u)|^2\eta^2|\nabla u|^2\,dx \\
&\quad\le \frac{1}{4\varepsilon}\int_\Omega |G(u)|^2|\nabla\eta|^2\,dx+\int_\Omega a(x)uF(u)\eta^2\,dx+d\int_\Omega F(u)\eta^2\,dx.
\end{aligned}
\tag{2.6}
$$

Next consider

$$
\begin{aligned}
\int_\Omega |\nabla(G(u)\eta)|^2\,dx &= \int_\Omega |G'(u)|^2\eta^2|\nabla u|^2\,dx \\
&\quad+2\int_\Omega G'(u)G(u)\eta\nabla u\cdot\nabla\eta\,dx+\int_\Omega |G(u)|^2|\nabla\eta|^2\,dx.
\end{aligned}
\tag{2.7}
$$

In view of (2.5) to (2.7), we take $\varepsilon>0$ small enough in (2.6) to get

$$
\begin{aligned}
&\int_\Omega |\nabla(G(u)\eta)|^2\,dx \\
&\quad\le C\left[\int_\Omega |G(u)|^2|\nabla\eta|^2\,dx+\int_\Omega |a(x)||G(u)|^2\eta^2\,dx+d\int_\Omega |F(u)|\eta^2\,dx\right],
\end{aligned}
\tag{2.8}
$$

where $C$ is a constant independent of $\delta$. Let $S>0$ be the best constant for the Sobolev embedding inequality on $H_0^1(\Omega)$, $S\|w\|_{2^*}^2\le\|\nabla w\|_2^2$. Now choose $\delta$ so small that

$$\|a(x)\|_{L^{n/2}(B(x_0,\delta))}\le\frac{S}{8C}$$

and

$$\frac{1}{4}S\left[\int_\Omega |v|^{p+1}\,dx\right]^{2/(p+1)}\le\int_\Omega |\nabla v|^2\,dx,$$

for all $v\in H^1(\Omega)$ supported in $B(x_0,\delta)\cap\overline{\Omega}$. We can eliminate the middle term on the right hand side of (2.7) to get

$$\int_\Omega |\nabla(G(u)\eta)|^2\,dx\le C\left[\int_\Omega |G(u)|^2|\nabla\eta|^2\,dx+d\int_\Omega |F(u)|\eta^2\,dx\right]. \tag{2.9}$$

Finally, we will iteratively use the estimate (2.9) to get the result. Firstly, we choose $\beta=[(n-1)/(n-2)]>1$ and notice that $u\in H^1(\Omega)\hookrightarrow L^{2n/(n-2)}(\Omega)\hookrightarrow L^{2\beta}(\Omega)$, which implies $\int_\Omega |G(u)|^2\,dx$ and $\int_\Omega |F(u)|\,dx<\infty$. Letting $N\to\infty$ in (2.9), we have $|u|^\beta\eta\in H^1(\Omega)$, so $|u|^\beta\in H^1(\Omega)\hookrightarrow L^{2n/(n-2)}(\Omega)$ from the arbitrariness of $x_0\in\overline{\Omega}$.

Secondly, choose $\beta = \beta_2 = [(n-1)/(n-2)]^2$ and notice that $L^{[2n/(n-2)]\beta}(\Omega) \hookrightarrow L^{2\beta_2}(\Omega)$, which implies $\int_\Omega |G(u)|^2\, dx$ and $\int_\Omega |F(u)|\, dx < \infty$. Letting $N \to \infty$ in (2.9) again, we get $|u|^{\beta_2}\eta \in H^1(\Omega)$, so $|u|^{\beta_2} \in H^1(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega)$ from the arbitrariness of $x_0 \in \overline{\Omega}$.

We then iteratively set $\beta = \beta_k = [(n-1)/(n-2)]^k$ to get $u^{\beta_k} \in H^1(\Omega)$ for all $k \in \mathbb{N}$. Therefore $u \in L^q(\Omega)$ for all $q > 1$. Thus we have verified $(i)$. On the other hand, $(ii)$ is simply a result taken from [9], we shall omit the proof here. $\square$

The second regularity lemma can also be found in [9].

**Lemma 2.1.3.** *Let $h(x)$ be in $W^{1,q}_{loc}(\Omega)$. If $u \in W^{2,p}_{loc}(\Omega)$ is a strong solution of* (2.2), *where $p, q \in (1, \infty)$, then $u \in W^{3,q}_{loc}(\Omega)$.*

**Remark 2.1.4.** By a strong solution of (2.2), we mean a twice weakly differentiable function $u$ satisfying $-\Delta u = h(x) + d$ almost everywhere in $\Omega$.

**Theorem 2.1.5.** *Suppose $\partial\Omega$ is $C^2$ and $p \in (1, (n+2)/(n-2)]$. Let $u \in X$ be a critical point to the energy functional $J$ given in* (1.7) *with $g(x,z) = |z|^{p-1}z + f(x,z)$ where $f(x,z)$ in $C^1(\overline{\Omega} \times \mathbb{R})$ satisfies the conditions:*
*For every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\begin{aligned} |f(x,z)| &\leq \varepsilon|z|^p + C_\varepsilon|z| \\ |\nabla f(x,z)| &\leq \varepsilon|z|^{p-1} + C_\varepsilon. \end{aligned} \tag{2.10}$$

*Then $u$ is a classical solution to* (1.5).

*Proof of Theorem* 2.1.5. We first show that every critical point is in $C^1(\overline{\Omega}) \cap C^2(\Omega)$. Let $u \in X$ be a critical point of $J$. For any $\varphi \in X$, we have

$$J'(u)\varphi = \int_\Omega \nabla u \cdot \nabla\varphi\, dx - \int_\Omega g(x, u+A)\varphi\, dx = 0.$$

For any $\psi \in H^1(\Omega)$, we have $\psi - \fint_\Omega \psi \in X$, hence

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla \psi \, dx &= \int_\Omega \nabla u \cdot \nabla\left(\psi - \fint_\Omega \psi \, dx\right) dx \\ &= \int_\Omega g(x, u+A)\left(\psi - \fint_\Omega \psi \, dx\right) dx \\ &= \int_\Omega \left(g(x,u+A) - \fint_\Omega g(x,u+A)\, dx\right) \psi \, dx.\end{aligned} \tag{2.11}$$

Denote $w := u + A$ and notice that $g(x,w) = |w|^{p-1}w + f(x,w)$. (2.11) becomes

$$\int_\Omega \nabla w \cdot \nabla \psi \, dx = \int_\Omega \left(g(x,w) - \fint_\Omega g(x,w)\, dx\right) \psi \, dx. \tag{2.12}$$

Then apply the regularity Lemma 2.1.1 to $w$ with $d := -\fint_\Omega g(x,w)\, dx$ and $a(x) := |w|^{p-1} + w^{-1}f(x,w)$, to conclude that $w \in L^q(\Omega)$ for all $q > 1$. Note one can easily obtain $a(x) \in L^{n/2}(\Omega)$, after using (2.10) and the Sobolev embedding theorem.

From (2.10), we further have $g(x,w) \in L^r(\Omega)$ for all $r > 1$. Now apply the regularity Lemma 2.1.1 with $d := -\fint_\Omega g(x,w)\, dx$ and $h(x) = g(x,w) \in L^r(\Omega)$, to conclude that $w \in W^{2,r}(\Omega)$ for all $r$. The Sobolev embedding theorem implies $w \in C^{1,\alpha}(\overline{\Omega})$ for every $\alpha \in (0,1)$.

To improve the smoothness to $C^2$, we want to apply Lemma 2.1.3 with $h(x) = g(x,w)$ as above when $w$ is a strong solution. Thus we need to show that $w$ is a strong solution to $-\Delta w = h(x) + d$ and that $h(x)$ is $W^{1,q}_{loc}(\Omega)$ for some large enough $1 < q < \infty$ such that $W^{3,q}_{loc}(\Omega) \hookrightarrow C^{2,\beta}(\Omega)$ for some $\beta \in (0,1)$.

To show that $w$ is a strong solution to the equation, we pick a sequence $\{w_m\}_{m=1}^\infty \subset C^\infty(\overline{\Omega})$ such that $\|w_m - w\|_{2,2} \to 0$. From the smoothness of $w_m$, we have

$$\int_\Omega -\Delta w_m \eta \, dx = \int_\Omega \nabla w_m \cdot \nabla \eta \, dx$$

for those $\eta \in C^\infty(\overline{\Omega})$ with support compactly contained in $\Omega$. We let $m \to \infty$ and conclude from $w \in W^{2,2}_{loc}(\Omega) \hookrightarrow W^{2,2}(spt(\eta))$ that

$$\begin{aligned}\int_\Omega -\Delta w \eta \, dx &= \int_\Omega \nabla w \cdot \nabla \eta \, dx \\ &= \int_\Omega \left(h - \fint_\Omega h \, dx\right) \eta \, dx = \int_\Omega (h+d)\eta \, dx.\end{aligned} \tag{2.13}$$

As $\eta$ is arbitrary, (2.13) implies that $-\Delta w = h - \fint_\Omega h\, dx$ almost everywhere in $\Omega$. In other words, $w$ is a strong solution to $-\Delta w = h(x) + d$.

Concerning the regularity of $h$, we assert that $h$ is $W^{1,q}_{loc}(\Omega)$ for all $q \geq 1$. By definition of $h$, we have

$$\frac{\partial h}{\partial x_k} = p|w|^{p-1}\frac{\partial w}{\partial x_k} + \frac{\partial f}{\partial x_k}(x,w) + \frac{\partial f}{\partial z}(x,w)\frac{\partial w}{\partial x_k},$$

for $k = 1, 2, \ldots, n$. Notice that $w \in W^{2,2}_{loc}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$ thus $|\nabla w| \in C^{0,\alpha}(\overline{\Omega}) \hookrightarrow L^\infty(\Omega)$. Together with (2.10) and the fact that $w \in L^s(\Omega)$ for all $s \geq 1$, we conclude that $|\nabla h| \in L^s(\Omega)$ for all $s \geq 1$. Recall that $h \in L^r(\Omega)$ for all $r \geq 1$. We consequently have $h \in W^{1,q}(\Omega)$ for all $q \geq 1$. Hence we conclude that $w \in W^{3,q}_{loc}(\Omega) \hookrightarrow C^{2,\beta}(\Omega) \hookrightarrow C^2(\Omega)$. Recall that $w \in C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ and thus have $u = w - A \in C^1(\overline{\Omega}) \cap C^2(\Omega)$.

We finally show that $u$ is a classical solution to (1.5) with the aid of the regularity of $u$ above. Perform integration by parts for any $\varphi \in C^\infty(\overline{\Omega})$ to get

$$\begin{aligned}\int_\Omega -\Delta u\varphi\, dx &= \int_\Omega \nabla u \cdot \nabla\varphi\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n}\varphi\, dS \\ &= \int_\Omega \left(h - \fint_\Omega h\, dx\right)\varphi\, dx - \int_{\partial\Omega}\frac{\partial u}{\partial n}\varphi\, dS.\end{aligned}$$

As $\varphi$ is arbitrary, we have $-\Delta u = h - \fint_\Omega h\, dx$ on $\Omega$, which implies

$$\int_{\partial\Omega}\frac{\partial u}{\partial n}\varphi\, dS = 0,$$

for all $\varphi \in C^\infty(\overline{\Omega})$. By the arbitrariness of $\varphi$ again, we conclude that $\partial u/\partial n = 0$ on $\partial\Omega$. This proof is completed. □

## 2.2 Regularity for $0 < p < 1$

In this section, we prepare the regularity result for Problem (1.5) with sublinear nonlinearity which will be discussed in the next chapter.

**Theorem 2.2.1.** *Suppose $\partial\Omega$ is $C^2$. Let $u \in X$ be a critical point to the energy functional $J$ given in* (1.7) *with $g(x,z) \in C^1(\overline{\Omega} \times \mathbb{R})$ satisfying $|g(x,z)| \leq C\,(1+|z|^\gamma)$, for some $\gamma \in (0,1)$. Then $u$ is a classical solution to* (1.5).

*Proof.* We first show that $u \in W^{2,r}(\Omega)$ for some $r > n$. Notice that by the Sobolev Embedding Theorem $u \in X \hookrightarrow L^{2^*}(\Omega)$. If $2^* > n$ then we are done. Otherwise, by the sublinear growth control of $g(x,z)$, we know that $g(x,u) \in L^{2^*/\gamma}(\Omega)$ together with Lemma 2.1.1, we have $u \in W^{2,(2^*/\gamma)}(\Omega)$. If $(2^*/\gamma) > n$, then we are done. Otherwise, utilize the growth control again to obtain $g(x,u) \in L^{(2^*/\gamma)^*/\gamma}(\Omega) \hookrightarrow L^{2^*/\gamma^2}(\Omega)$, which implies $u \in W^{2,(2^*/\gamma^2)}(\Omega)$, by Lemma 2.1.1. Iterating the above argument, we have $u \in W^{2,r}(\Omega)$ for some $r > n$.

Since $u \in X$ is a critical point of $J$, by (1.8) we have

$$\int_\Omega |\nabla u|^2\, dx = \int_\Omega g(x,u+A)u\, dx.$$

Together with the growth control of $g(x,z)$ and the Poincare inequality, we have

$$\begin{aligned}
\|\nabla u\|_2^2 =& \int_\Omega |\nabla u|^2\, dx \leq C\int_\Omega |u|\, dx + C\int_\Omega |u|^\gamma\, dx \\
=& C\|u\|_1 + C\|u\|_{\gamma+1}^{\gamma+1} \leq C\|u\|_2 + C\|u\|_2^{\gamma+1} \\
\leq& C\|\nabla u\|_2 + C\|\nabla u\|_2^{\gamma+1},
\end{aligned}$$

where $C$ is a generic constant depending only on $n$, $\Omega$, $\gamma$ and $A$. In this proof, we will repeatedly use the constant $C$ in the same way. Then we obtain a constant $R_1$ independent of the choice of $u$ such that $\|u\|_1 \leq C\|\nabla u\|_2 \leq R_1$. Thus Lemma 2.1.1 implies

$$\begin{aligned}
\|u\|_{2,r} \leq& C\|g(x,u)\|_r + C\|u\|_1 \\
\leq& C\left(1 + \||u|^\gamma\|_\gamma\right) + CR_1 \\
\leq& C\left(1 + \|u\|_{\gamma r}^\gamma\right) + CR_1 \leq C + C\|u\|_{2,r}^\gamma
\end{aligned}$$

This implies the existence of a constant $R_2 > 0$ independent of the choice of $u$ such that $\|u\|_{2,r} \leq R_2$. Compactness of the embedding $W^{2,r}(\Omega) \hookrightarrow C^1(\overline{\Omega})$

ensures the existence of another constant $R_3 > 0$ such that $\|u\|_{C^1(\Omega)} \leq R_3$. Therefore $g(x,u) \in C^1(\overline{\Omega}) \hookrightarrow W^{1,r}(\Omega)$. Finally, Lemma 2.1.3 implies that $u \in W^{3,r}_{loc}(\Omega) \hookrightarrow C^2(\Omega)$ and hence $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. By a similar argument as in the last paragraph of the proof of Theorem 2.1.5, we conclude that $u$ is a classical solution. $\square$

# Chapter 3

# The Sublinear Case

In this chapter we deal with the following problem

$$\begin{cases} -\Delta u = g(x,u) - \fint_\Omega g(x,u)\, dx & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \fint_\Omega u\, dx = A, \end{cases} \tag{3.1}$$

where $A$ is a real number and $g(x,z)$ is in $C^1(\overline{\Omega} \times \mathbb{R})$ satisfying

$$|g(x,z)| \le C(1 + |z|^\gamma), \tag{3.2}$$

for some $\gamma \in (0,1)$. Hereafter in this chapter, the generic constant $C$ will be used to denote a positive constant depending only on $n$, $\Omega$, $\gamma$ and $A$. We are going to prove the existence of a solution to Problem (3.1) via minimization.

We shall first transform the problem into the form of (1.5) and use minimization with the aid of Theorem 2.2.1 to find a classical solution.

## 3.1 Existence of a minimizer

By a standard direct method argument, we first prove a coercive inequality.

**Lemma 3.1.1.** *There exist two constants $C_1$ and $C_2 > 0$ such that for all $w \in X$, where $X$ is given in* (1.6), *we have*

$$J(w) \geq C_1 \|w\|_2^2 - C_2. \tag{3.3}$$

*Proof.* Fix any $w \in X$. By the definition of $J$ in (1.7) and (3.2), we have

$$\begin{aligned} J(w) \geq & \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx - C\int_\Omega 1 + |w+A|^{\gamma+1} \, dx + \int_\Omega G(x,A) \, dx \\ \geq & \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx - C\int_\Omega |w|^{\gamma+1} \, dx + \left(\int_\Omega G(x,A) \, dx - C|\Omega| - C\right) \\ \geq & \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx - C\int_\Omega |w|^{\gamma+1} \, dx - C \end{aligned}$$

where $G(x,z) := \displaystyle\int_0^z g(x,t) \, dt$. By the Jensen inequality and the Sobolev inequality, we have

$$\begin{aligned} J(w) \geq & \frac{1}{2}\|\nabla w\|_2^2 - C\left(\int_\Omega |\nabla w|^2 \, dx\right)^{(\gamma+1)/2} - C \\ \geq & \frac{1}{2}\|\nabla w\|_2^2 - C\|\nabla w\|_2^{\gamma+1} - C \\ \geq & \frac{1}{4}\|\nabla w\|_2^2 - C. \end{aligned}$$

□

With this inequality, we move on to find a global minimizer by a compactness argument.

**Theorem 3.1.2.** *Problem* (3.1) *admits a classical solution.*

*Proof.* In view of Theorem 2.2.1, every critical point to the energy $J$ given in (1.7) is a classical solution. Hence all we need is to prove that the energy $J$ admits a critical point, more precisely, a global minimum.

To see this, we first note that by the coercive inequality, $J$ is bounded from below and hence we have $m := \inf_{w \in X} J(w) > -\infty$. Let $\{u_j\}_{j=1}^\infty \subset X$ be a minimizing sequence, that is,

$$J(u_j) \to m, \tag{3.4}$$

as $j \to \infty$. By coercivity again, we have for large enough $j$

$$C_1\|\nabla u_j\|_2^2 \le J(u_j) + C_2 \le m + 1 + C_2$$

and therefore $\|u_j\|_2$ is bounded. We can now employ the Rellich compact embedding theorem and the weak compactness property of reflexive Banach spaces to extract a subsequence which we may still call $\{u_j\}_{j=1}^{\infty}$ so that there exists $u \in X$ such that

$$\begin{cases} u_j \to u & \text{weakly in } X \\ u_j \to u & \text{strongly in } L^r, \text{ for all } 1 \le r < 2^* \\ u_j \to u & \text{almost everywhere in } \Omega. \end{cases} \tag{3.5}$$

By the convexity of $w \mapsto \|\nabla w\|_2^2$, we have

$$\begin{aligned} J(u_j) =& \frac{1}{2}\int_\Omega |\nabla u_j|^2 \, dx - \int_\Omega G(x, u_j + A) - G(x, A) \, dx \\ \ge& \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \int_\Omega \nabla u \cdot \nabla(u_j - u) \, dx \\ &- \int_\Omega G(x, u_j + A) - G(x, A) \, dx. \end{aligned} \tag{3.6}$$

By the weak convergence in (3.5), we have

$$\int_\Omega \nabla u \cdot \nabla(u_j - u) \, dx \to 0. \tag{3.7}$$

By the strong convergence in (3.5) and the growth control of $g(x,z)$, we have

$$\begin{aligned} &\left|\int_\Omega G(x, u_j + A) - G(x, u + A) \, dx\right| \\ =& \left|\int_\Omega g(x, \theta(u_j - u) + u + A)(u_j - u) \, dx\right| \\ \le& C\int_\Omega (1 + |\theta(u_j - u) + u + A|^\gamma) \, |u_j - u| \, dx \\ \le& C\int_\Omega (1 + |u + A|^\gamma)|u_j - u| + |u_j - u|^{\gamma+1} \, dx \\ \le& C\|1 + |u + A|^\gamma\|_2 \cdot \|u_j - u\|_2 + C\|u_j - u\|_{\gamma+1}^{\gamma+1}, \end{aligned}$$

which implies

$$\int_\Omega G(x, u_j + A) \, dx \to \int_\Omega G(x, u + A) \, dx. \tag{3.8}$$

With (3.7) and (3.8), we let $j \to \infty$ in (3.6) and obtain $m \geq J(u)$. Consequently, $u \in X$ is a minimizer of $J$ in $X$. $\square$

## 3.2 Uniqueness and multiplicity

In the last section we have shown that Problem (3.1) admits a solution. Now we state a uniqueness result.

**Proposition 3.2.1.** *Consider Problem* (3.1). *Suppose* $(\partial g/\partial z) \leq \alpha$ *on* $\overline{\Omega} \times \mathbb{R}$, *for some* $\alpha < \mu_1$, *where* $\mu_1$ *is the first eigenvalue to the eigenvalue problem* $(B.1)$. *Then it has a uniqueness solution.*

*Proof.* It suffices to show uniqueness. Let $u$ and $v$ be two solutions to Problem (3.1), that is,

$$\begin{cases} -\Delta u = g(x,u) - \fint_\Omega g(x,u)\ dx, \\ -\Delta v = g(x,v) - \fint_\Omega g(x,v)\ dx. \end{cases}$$

This implies

$$\begin{aligned} -\Delta(u-v) =& [g(x,u) - g(x,v)] - \fint_\Omega (g(x,u) - g(x,v))\ dx \\ =& \frac{\partial g}{\partial z}(x, v + \theta(u-v))(u-v) - \fint_\Omega (g(x,u) - g(x,v))\ dx. \end{aligned}$$

Notice that $\int_\Omega (u-v)\ dx = 0$ and $\partial(u-v)/\partial n = 0$ on $\partial\Omega$, we have

$$\begin{aligned} \int_\Omega |\nabla(u-v)|^2\ dx =& \int_\Omega -[\Delta(u-v)](u-v)\ dx \\ =& \int_\Omega \frac{\partial g}{\partial z}(x, v+\theta(u-v))(u-v)^2\ dx \\ \leq& \alpha \int_\Omega (u-v)^2\ dx \\ \leq& \frac{\alpha}{\mu_1} \int_\Omega |\nabla(u-v)|^2\ dx, \end{aligned} \tag{3.9}$$

which implies $\int_\Omega |\nabla(u-v)|^2\ dx \leq 0$ and we have $u \equiv v$ on $\Omega$. $\square$

Suppose now $g(x,z)$ is independent of $x$, that is, $g(x,z) = g(z)$. We see that (3.1) has $u_c \equiv A$ as its constant solution.

Recall that in the proof of Theorem 3.1.2, we have shown that the equation (3.1) has a solution which is a minimizer of $J$. In view of the minimization problem, the constant solution $u_c$ above corresponds to the trivial solution $u_0 \equiv 0$. If $u_0$ is a minimizer of the energy $J$, we would get no further information. But if $u_0$ is NOT a minimizer, we then obtain another solution $u$ from minimization. We have the following multiplicity result.

**Proposition 3.2.2.** *Suppose $g = g(z)$ in Problem (3.1). If $g'(A) > \mu_1$, where $\mu_1$ is the first eigenvalue to the eigenvalue problem $(B.1)$, then $u_0 \equiv 0$ is not a minimizer and hence Problem (3.1) has a nontrivial solution.*

*Proof.* If $u_0 \equiv 0$ is a minimizer, then for any $\varphi \in X$, $d^2/dt^2 \,|_{t=0}\, [J(0 + t\varphi)] \geq 0$ which implies

$$\int_\Omega |\nabla \varphi|^2 \, dx - \int_\Omega g'(A)\varphi^2 \, dx \geq 0.$$

Let $\varphi_1$ be an eigenfunction correspond to the eigenvalue $\mu_1$. Then

$$\int_\Omega |\nabla \varphi_1|^2 \, dx - \int_\Omega g'(A)\varphi_1^2 \, dx = (\mu_1 - g'(A)) \int_\Omega \varphi_1^2 \, dx < 0.$$

which is a contradiction. $\square$

# Chapter 4

# The Superlinear-Subcritical Case I

In this chapter we deal with the following problem

$$\begin{cases} -\Delta u = |u|^{p-1}u + f(x) - \fint_\Omega \left(|u|^{p-1} + f(x)\right)\, dx & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \fint_\Omega u \, dx = A, & \end{cases} \tag{4.1}$$

where $A$ is a real number, $1 < p < \min\{2, (n+2)/(n-2)\}$ and $f(x)$ is a non-constant $L^2$ function on $\Omega$. As we have mentioned in Section 1.3, since $f$ is not a constant function, problem (4.1) has no constant (trivial) solution. We are interested in the existence of solution in this case. We use the technique of Nehari manifold to produce the following existence result.

**Theorem 4.0.1.** *There exist positive constants $A_0$ and $K_0$ such that for each $A \in (-A_0, A_0)$ and $f \in L^2(\Omega)$ with $\|f\|_2 \in (0, K_0)$, Problem (4.1) admits a classical solution.*

We shall transform the problem into the form of (1.5) and use calculus of variations to find a solution. We consider minimization on a special submanifold of the space $X$, namely, the Nehari manifold which we will define later. After that, we show that the minimizer is in fact a critical point of the energy in the whole space $X$. In this chapter, the generic constant $C$ will be used to denote a positive constant depending only on $n$, $\Omega$ and $p$.

## 4.1 The Nehari manifold

In this case the energy $J$ defined in (1.7) reads as

$$J(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx - \frac{1}{p+1}\int_\Omega \left(|w+A|^{p+1} - |A|^{p+1}\right) \, dx - \int_\Omega f(x) w \, dx \quad (4.2)$$

for $w \in X$. Then

$$J'(w)\varphi = \int_\Omega \nabla w \cdot \nabla \varphi \, dx - \int_\Omega |w+A|^{p-1}(w+A)\varphi \, dx - \int_\Omega f(x)\varphi \, dx \quad (4.3)$$

for $w, \varphi \in X$. We further define for $w \in X$

$$G(w) := J'(w)w = \int_\Omega |\nabla w|^2 \, dx - \int_\Omega |w+A|^{p-1}(w+A)w \, dx - \int_\Omega f(x) w \, dx. \quad (4.4)$$

We now introduce the Nehari manifold $M := \{w \in X : G(w) = 0\}$. To show that $M$ is a manifold in $X$, we need the following lemma.

**Lemma 4.1.1.** *There exist positive constants $A_0$ and $K_0$ such that for every $A \in (-A_0, A_0)$ and $f \in L^2(\Omega)$ with $\|f\|_2 \in (0, K_0)$, we have $G'(w)w \neq 0$ for $w \in M \setminus \{0\}$.*

*Proof.* Differentiate $G$ to get for $v$, $\varphi \in X$

$$\begin{aligned} G'(v)\varphi =& 2\int_\Omega \nabla v \cdot \nabla \varphi \, dx - (p+1)\int_\Omega |v+A|^{p-1}(v+A)\varphi \, dx \\ &+ Ap\int_\Omega |v+A|^{p-1}\varphi \, dx - \int_\Omega f(x)\varphi \, dx. \end{aligned} \quad (4.5)$$

If $G'(w)w = 0$, for some $w \in M \setminus \{0\}$, we have

$$\begin{aligned} 0 =& 2\int_\Omega |\nabla w|^2 \, dx - (p+1)\int_\Omega |w+A|^{p+1} \, dx \\ &+ A(p+1)\int_\Omega |w+A|^{p-1}(w+A) \, dx \\ &+ Ap\int_\Omega |w+A|^{p-1} w \, dx - \int_\Omega f(x) w \, dx. \end{aligned} \quad (4.6)$$

As $w \in M$ we have

$$\int_\Omega |\nabla w|^2 \, dx - \int_\Omega |w+A|^{p-1}(w+A)w \, dx - Ap \int_\Omega |w+A|^{p-1} w \, dx = 0. \quad (4.7)$$

Subtracting (4.7) from (4.6) and (4.6) from $(p+1)$ times (4.7) respectively, we have

$$\int_\Omega |\nabla w|^2 \, dx - p\int_\Omega |w+A|^{p-1}(w+A)w \, dx - \int_\Omega f(x) w \, dx = 0, \quad (4.8)$$

and

$$(p-1)\int_\Omega |\nabla w|^2 \, dx - p\int_\Omega f(x) w \, dx - Ap \int_\Omega |w+A|^{p-1} w \, dx = 0. \quad (4.9)$$

From (4.8) and Proposition $A.1$, we have

$$\begin{aligned}
&\int_\Omega |\nabla w|^2 \, dx \\
=&p\int_\Omega |w+A|^{p+1} \, dx - Ap\int_\Omega |w+A|^{p-1}(w+A) \, dx - Ap\int_\Omega |w+A|^{p-1} w \, dx \\
=&p\int_\Omega \left(|w+A|^{p+1} - |A|^{p+1}\right) \, dx - Ap\int_\Omega |w+A|^{p-1}(w+A) - |A|^p \, dx \\
&- Ap\int_\Omega |w+A|^{p-1} w \, dx \\
\leq& p\int_\Omega |w|^{p+1} \, dx + \frac{(p+1)p}{2}|A|^{p-1}\int_\Omega |w|^2 \, dx - Ap^2\int_\Omega |\theta w + A|^{p-1} w \, dx \\
&- Ap\int_\Omega |w+A|^{p-1} w \, dx \\
\leq& C\|w\|_{p+1}^{p+1} + C|A|^{p-1}\int_\Omega |w|^2 \, dx + (C|A|)\, \|\theta w + A\|_{p+1}^{p-1} \cdot \|w\|_{p+1}^2 \\
&+ (C|A|)\, \|w+A\|_{p+1}^{p-1} \cdot \|w\|_{p+1}^2 \\
\leq& C\|w\|_{p+1}^{p+1} + C|A|^{p-1}\int_\Omega |w|^2 \, dx + (C|A|)\, \|w\|_{p+1}^{p+1} + (C|A|^p)\, \|w\|_{p+1}^2 \\
&+ (C|A|)\, \|w\|_{p+1}^{p+1} + (C|A|^p)\, \|w\|_{p+1}^2 \\
\leq& (C + C|A|)\, \|w\|_{p+1}^{p+1} + \left(C|A|^{p-1}\right) \|w\|_2^2 + (C|A|^p)\, \|w\|_{p+1}^2.
\end{aligned}$$

By Jensen inequality, Sobolev inequality and Poincare inequality, we have

$$(1 - C|A|^{p-1} - C|A|^p)\|\nabla w\|_2^2 \leq (C + C|A|)\, \|\nabla w\|_2^{p+1}.$$

As we $w \not\equiv 0$, we may eliminating $\|\nabla w\|_2^2$ from both sides and get

$$(1 - C|A|^{p-1} - C|A|^p) \leq (C + C|A|)\,\|\nabla w\|_2^{p-1}.$$

Thus there is positive a constant $\alpha$ depending only on $n$, $\Omega$ and $p$, such that for all $A$ with $|A|$ small enough, we have

$$\|\nabla w\|_2 \geq \alpha > 0. \tag{4.10}$$

On the other hand, from (4.9) we have

$$\begin{aligned}
(p-1)\|\nabla w\|_2^2 =&(p-1)\int_\Omega |\nabla w|^2\,dx\\
=&p\int_\Omega f(x)w\,dx + Ap\int_\Omega |w+A|^{p-1}w\,dx\\
\leq&C\|f\|_2\cdot\|w\|_2 + (p|A|)\,\|w+A\|_p^{p-1}\cdot\|w\|_p\\
\leq&C\|f\|_2\cdot\|w\|_2 + (p|A|)\,\|w\|_p^p + (p|A|^p)\,\|w\|_p\\
\leq&C\|f\|_2\cdot\|\nabla w\|_2 + (C|A|)\,\|\nabla w\|_2^p + (C|A|^p)\,\|\nabla w\|_2.
\end{aligned}$$

Again $w \not\equiv 0$, we eliminate $\|\nabla w\|_2$ from from both sides and obtain

$$\|\nabla w\|_2 \leq C\|f\|_2 + C|A|^p + (C|A|)\,\|\nabla w\|_2^{p-1}.$$

As $0 < p-1 < 1$, there exists a constant $C > 0$ such that for all $z \in \mathbb{R}$ we have $|z|^{p-1} \leq C + |z|$. Consequently,

$$\|\nabla w\|_2 \leq C\|f\|_2 + C|A|^p + C|A| + C|A|\|\nabla w\|_2.$$

Thus for those $A$ with $|A|$ small enough, we have

$$\|\nabla w\|_2 \leq C\|f\|_2 + C|A|^p + C|A|.$$

Together with (4.10), we obtain

$$0 < \alpha \leq \|\nabla w\|_2 \leq C\|f\|_2 + C|A|^p + C|A|.$$

Therefore, when $|A|$ is small enough,

$$\|f\|_2 \geq \frac{\alpha}{2} =: K_0 > 0.$$

Thus the contrapositive statement of the lemma is verified and we are done. $\square$

We readily see that the Nehari manifold defined above is a $C^1$-manifold.

**Proposition 4.1.2.** *Let $A_0$ and $K_0$ be two positive constants given in Lemma (4.1.1). Then for each $A \in (-A_0, A_0)$ and $f \in L^2(\Omega)$ with $\|f\|_2 \in (0, K_0)$, $M := \{w \in X : G(w) = 0\}$ is a $C^1$-manifold.*

*Proof.* From Lemma 4.1.1, we know that for all $w \in M \setminus \{0\}$, $G'(w)w \neq 0$, which implies $G'(w) \not\equiv 0$. As $0 \in M$, we need to show that $G'(0) \not\equiv 0$. By (4.5), we have

$$G'(0)\varphi = -\int_\Omega f(x)\varphi \, dx,$$

for $\varphi \in X$. As $f$ is not a constant function, we conclude that $G'(0) \not\equiv 0$. Now, the desired result follows from the implicit function theorem. □

## 4.2 Minimization on the Nehari manifold

We begin with the following proposition.

**Proposition 4.2.1.** *Let $A$ and $f$ be as in Proposition* 4.1.2, *we have $J_M := J|_M$ is bounded below on $M$.*

*Proof.* Fix any $u$ in $M$. Combining (4.6) and (4.7) we have

$$\begin{aligned} J_M(w) =& \frac{1}{n}\int_\Omega |\nabla w|^2 \, dx - \frac{A}{p+1}\int_\Omega |w+A|^{p-1}(w+A) - |A|^{p-1}A \, dx \\ &- \frac{p}{p+1}\int_\Omega f(x) w \, dx. \end{aligned} \tag{4.11}$$

Notice that

$$\begin{aligned} \int_\Omega |w+A|^{p-1}(w+A) \, dx \leq& \|w+A\|_p^p \leq C\|w\|_p^p + C|A|^p \\ \leq& C\|\nabla w\|_2^p + C|A|^p \end{aligned}$$

and that

$$\int_\Omega f(x) w \, dx \leq \|f\|_2 \cdot \|w\|_2 \leq C\|f\|_2 \cdot \|\nabla w\|_2.$$

Thus from (4.11) we get

$$\begin{aligned} J_M(w) \geq & C\|\nabla w\|_2^2 - (C|A|)\,\|\nabla w\|_2^p - C\|f\|_2 \cdot \|\nabla w\|_2 - C|A|^p \\ \geq & C\|\nabla w\|_2^2 - (C|A_0|)\,\|\nabla w\|_2^p - CK_0 \cdot \|\nabla w\|_2 - C|A_0|^p. \end{aligned} \tag{4.12}$$

As $p \in (1,2)$, we see that $J$ is bounded below by a constant independent of $w$. $\square$

We next show the existence of minimizer by direct method.

**Proposition 4.2.2.** *Let $A$ and $f$ be as in Proposition 4.1.2, we have $J_M := J|_M$ attains its minimum on $M$.*

*Proof.* Since $J_M$ is bounded below, we may set $m := \inf_{w \in M} J_M(w)$. Let $\{u_j\}_{j=1}^{\infty} \subset M$ be a minimizing sequence. Then for large enough $j$, $J_M(u_j) \leq m+1$. Rewrite (4.12) as

$$C\|\nabla u_j\|_2^2 \leq m + 1 + (C|A_0|)\,\|\nabla u_j\|_2^p + C\|\nabla u_j\|_2 + C|A_0|^p.$$

As $p \in (1,2)$, we conclude that $\|\nabla u_j\|_2$ is a bounded sequence. Then there exists some $u \in X$ such that by passing to a subsequence which we still call $\{u_j\}_{j=1}^{\infty}$, we have

$$\begin{cases} u_j \to u & \text{weakly in } X \\ u_j \to u & \text{strongly in } L^r(\Omega), \text{ for } 1 \leq r < 2^* \\ u_j \to u & \text{a.e. in } \Omega. \end{cases} \tag{4.13}$$

By the convexity of $w \mapsto \|\nabla w\|_2^2$, we have

$$\int_\Omega |\nabla u_j|^2\, dx \geq \int_\Omega |\nabla u|^2\, dx + 2\int_\Omega \nabla u \cdot \nabla(u_j - u)\, dx.$$

Combining this with (4.11), we have

$$\begin{aligned} J_M(u_j) \geq & \int_\Omega |\nabla u|^2\, dx + 2\int_\Omega \nabla u \cdot \nabla(u_j - u)\, dx \\ & - \frac{A}{p+1}\int_\Omega |u_j + A|^{p-1}(u_j + A) - |A|^{p-1}A\, dx - \frac{p}{p+1}\int_\Omega f(x)u_j\, dx. \end{aligned}$$

By (4.13), we get

$$\begin{aligned} m \geq & \int_\Omega |\nabla u|^2 \, dx - \frac{A}{p+1} \int_\Omega |u+A|^{p-1}(u+A) \\ & - |A|^{p-1} A \, dx - \frac{p}{p+1} \int_\Omega f(x) u_j \, dx \\ = & J_M(u) \geq m. \end{aligned}$$

Consequently, $u$ is a minimizer. □

## 4.3 Minimizer on the Nehari manifold as a critical point on the whole space

To guarantee that a minimizer or more generally a critical point on the Nehari manifold is indeed a critical point on the whole space $X$, we have the following proposition.

**Proposition 4.3.1.** *Let $A$ and $f$ be as in Proposition 4.1.2. If $w \in M$ is such that $G'(w)w \neq 0$ and that $J'_M(w) = 0$, then $J'(w) = 0$.*

*Proof.* Note that $\nabla G(w)$ is orthogonal to $T_w(M)$, the tangent space of $M$ at $w$. Let $\widehat{\nabla G(w)}$ be the unit vector of $\nabla G(w)$. Consider the decomposition of $w$ with respect to this vector

$$w = \langle \widehat{\nabla G(w)}, w \rangle \widehat{\nabla G(w)} + \left( w - \langle \widehat{\nabla G(w)}, w \rangle \widehat{\nabla G(w)} \right). \tag{4.14}$$

Noting that $w - \langle \widehat{\nabla G(w)}, w \rangle \widehat{\nabla G(w)}$ lies in $T_w(M)$,

$$\begin{aligned} & J'(w) \left( w - \langle \widehat{\nabla G(w)}, w \rangle \widehat{\nabla G(w)} \right) \\ = & J'_M(w) \left( w - \langle \widehat{\nabla G(w)}, w \rangle \widehat{\nabla G(w)} \right) = 0. \end{aligned} \tag{4.15}$$

As $\|\nabla G(w)\|_X \langle \widehat{\nabla G(w)}, w \rangle = G'(w)w \neq 0$ and $J'(w)w = G(w) = 0$, we conclude from (4.14) and (4.15) that

$$J'(w)(\nabla G(w)) = 0,$$

whence $J'(w) = 0$. □

In the previous section, we have found a minimizer on the Nehari manifold in the proof of Proposition 4.2.2, that means $u \in X$ such that $J'_M(u) = 0$. In Lemma 4.1.1, we see that if this $u$ is nonzero, then $G'(u)u \neq 0$ and hence the condition of Proposition 4.3.1 is satisfied and hence $u$ is a global critical point. Therefore, all we need is to show that the minimizer $u$ on the Nehari manifold is not zero.

To see this, Let $w$ be in $M$. Then from (4.2) and (4.8) we have

$$\begin{aligned}
J_M(w) =& -\frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \frac{1}{p+1}\int_\Omega \left(|w+A|^{p+1} - |A|^{p+1}\right)\,dx \\
&+ \int_\Omega |w+A|^{p-1}(w+A)w\,dx \\
=& -\frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \frac{1}{p+1}\int_\Omega \left(|w+A|^{p+1} - |A|^{p+1}\right)\,dx \\
&+ \int_\Omega p|\theta w + A|^{p-1}w^2\,dx \\
\leq& -\frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \frac{1}{p+1}\int_\Omega |w|^{p+1}\,dx + \frac{p}{2}|A|^{p-1}\int_\Omega |w|^2\,dx \\
&+ p\int_\Omega |w|^{p+1}\,dx + p|A|^{p-1}\int_\Omega |w|^2\,dx \\
=& -\frac{1}{2}\|\nabla w\|_2^2 + \left(p - \frac{1}{p+1}\right)\int_\Omega |w|^{p+1}\,dx + \frac{3}{2}|A|^{p-1}\int_\Omega |w|^2\,dx \\
\leq& -\frac{1}{2}\|\nabla w\|_2^2 + C\|\nabla w\|_2^{p+1} + C|A|^{p-1}\|\nabla w\|_2^2 \\
=& -\frac{1}{2}\left(1 - C|A|^{p-1}\right)\|\nabla w\|_2^2 + C\|\nabla w\|_2^{p+1}.
\end{aligned}$$

We may further restrict $A_0 > 0$ smaller than that is given in Lemma 4.1.1 if necessary, so to guarantee that for all $A \in (-A_0, A_0)$ we have $1 - C|A|^{p-1} < 1/2$, and hence the above estimate reads as

$$J_M(w) \leq -\frac{1}{4}\|\nabla w\|_2^2 + C\|\nabla w\|_2^{p+1}.$$

As $0 \in M$, we may find $w \in M \setminus \{0\}$ with $\|\nabla w\|_2$ small enough such that $J_M(w) \leq -\frac{1}{8}\|\nabla w\|_2^2 < 0$, which implies $\min_{w \in M} J_M(w) < 0$ and hence $u$ is nonzero. The proof of Theorem 4.0.1 is completed.

# Chapter 5

# The Superlinear-Subcritical Case II

In this chapter we deal with the following problem

$$\begin{cases} -\Delta u = g(x,u) - \fint_\Omega g(x,u)\,dx & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \fint_\Omega u\,dx = A, \end{cases} \tag{5.1}$$

where $A$ is a real number and $g(x,z)$ is in $C^1(\overline{\Omega} \times \mathbb{R})$ with $g(x,A) \equiv$ constant. We see that there is always a trivial solution $u_c \equiv A$ and we are interested in nontrivial ones. Suppose $p \in (1,(n+2)/(n-2))$. We further assume that $g$ and $g'$ satisfy the following growth conditions:

(i) There exist positive constants $\lambda$ and $C_1$ such that for every $x \in \overline{\Omega}$ and $z \in \mathbb{R}$, we have

$$|g'(x,z)| \leq \lambda + C_1|z|^{p-1}. \tag{5.2}$$

(ii) There exist $r > 0$ and $\mu > 2$ such that for all $x$ in $\overline{\Omega}$ and $z$ with $|z| \geq r$, we have

$$0 < \mu G(x,z) \leq g(x,z)z. \tag{5.3}$$

Notice that by taking integration on the both sides of

$$\frac{g(x,z)}{G(x,z)} \geq \frac{\mu}{z}$$

and the continuity of $g$, we have the following consequence of (5.3).

$$G(x,z) \geq C|z|^{\mu} - C. \tag{5.4}$$

We are going to prove the following existence theorem with the aid of the mountain pass lemma.

**Theorem 5.0.1.** *There exist positive $\lambda_0$ and $A_0$ such that for each $\lambda \in (0, \lambda_0)$ and $A \in (-A_0, A_0)$, (5.1) has a nontrivial classical solution.*

Following the ideas in Section 1.2 and Theorem 2.1.5, we again transform the problem to a variational problem associated to the energy $J$. We first recall the standard version of the mountain pass lemma. After that, we apply it to prove the above theorem. As before, the generic constant $C$ in this chapter will be used to denote positive constants depending only on $n$, $\Omega$ and $p$.

## 5.1 The mountain pass lemma

We state the mountain pass lemma taken from [1]. The Palais-Smale condition will be stated in Proposition 5.2.3.

**Definition 5.1.1.** *Let $X$ be a real Banach space and $J$ be continuous functional on $X$. We say that $J$ satisfies the mountain pass property at a point $v \in X$ if*

*(i)* $J(v) = 0$,

*(ii) There are positive constants $\rho$ and $\alpha$ such that $J|_{\partial B_\rho(v)} \geq \alpha$, where $\partial B_\rho(v)$ is the set $\{w \in X : \|w - v\|_X = \rho\}$, and*

*(iii) There exist some $w_0$ in $X$ with $\|w_0 - v\|_X > \rho$ such that $J(w_0) \leq 0$.*

**Theorem 5.1.2.** *Let $X$ be a real Banach space and $J$ be $C^1$-functional on $X$ satisfying the Palais-Smale condition and the mountain pass property near the* *origin. Then $J$ has a critical value $c \geq a$ which can be characterized as*

$$c = \inf_{\gamma \in \Gamma} \max_{w \in \gamma} I(w) \tag{5.5}$$

*where $\Gamma$ is the class of all continuous paths from 0 to $w_0$ and $w_0$ is given by the mountain pass property (iii).*

It is clear that any critical point with the critical value $c$ given in Theorem 5.1.2 is nonzero.

## 5.2 Existence of a nontrivial critical point

To apply the mountain pass lemma, we verify that $J$ has a mountain pass near 0 and $J$ satisfies the Palais-Smale condition.

**Lemma 5.2.1.** *Let $\mu_1$ be the first eigenvalue of the eigenvalue problem (B.1). Then for all $\alpha \in \mathbb{R}$ with $0 < \alpha < \mu_1$ and $u \in X$, we have*

$$\int_{\Omega} |\nabla w|^2 \, dx - \alpha \int_{\Omega} |w|^2 \, dx \geq \frac{\mu_1 - \alpha}{\mu_1} \int_{\Omega} |\nabla w|^2 \, dx.$$

*Proof.* Note that for all $w \in X$, we have

$$\int_{\Omega} |\nabla w|^2 \, dx \geq \mu_1 \int_{\Omega} |w|^2 \, dx.$$

Therefore,

$$\begin{aligned}
&\int_{\Omega} |\nabla w|^2 \, dx - \alpha \int_{\Omega} |w|^2 \, dx \\
=&\frac{\mu_1 - \alpha}{\mu_1} \int_{\Omega} |\nabla w|^2 \, dx + \frac{\alpha}{\mu_1} \int_{\Omega} |\nabla w|^2 \, dx - \alpha \int_{\Omega} |w|^2 \, dx \\
\geq&\frac{\mu_1 - \alpha}{\mu_1} \int_{\Omega} |\nabla w|^2 \, dx + \frac{\alpha}{\mu_1} \left( \mu_1 \int_{\Omega} |w|^2 \, dx \right) - \alpha \int_{\Omega} |w|^2 \, dx \\
=&\frac{\mu_1 - \alpha}{\mu_1} \int_{\Omega} |\nabla w|^2 \, dx.
\end{aligned}$$

□

**Proposition 5.2.2.** *There is a positive constant $A_0$ such that for each $A \in (-A_0, A_0)$, $J$ satisfies the mountain pass property near the origin.*

*Proof.* $(i)$ is trivial. To prove $(ii)$, first notice that $G(x, w+A) - G(x, A) = g(x, A)w + \frac{1}{2}g'(x, \theta w + A)w^2$ and $w$ is of mean zero. Together with (5.2) we have

$$\begin{aligned}
J(w) =& \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx - \int_\Omega G(x, w+A) - G(x, A) \, dx \\
=& \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx - \int_\Omega \frac{1}{2} g'(x, \theta w + A) w^2 \, dx \\
\geq& \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx - \frac{1}{2}\int_\Omega \left(\lambda w^2 + C_1 |\theta w + A|^{p-1} w^2\right) dx \qquad (5.6)\\
\geq& \frac{1}{2}\left[\int_\Omega |\nabla w|^2 \, dx - (\lambda + k_p C_1 |A|^{p-1}) \int_\Omega w^2 \, dx\right] \\
&- k_p C_1 \int_\Omega |w|^{p+1} \, dx.
\end{aligned}$$

Choosing $\lambda_0$ and $A_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ and $A \in (-A_0, A_0)$, we have $\alpha := \lambda + k_p C_1 |A|^{p-1} < \mu_1$. Note that here $k_p$ is the least constant such that $|a+b|^{p+1} \leq k_p(|a|^{p+1} + |b|^{p+1})$ for all $a$ and $b \in \mathbb{R}$. By Lemma 5.2.1, we have

$$J(w) \geq \beta \int_\Omega |\nabla w|^2 \, dx - k_p C_1 \int_\Omega |w|^{p+1} \, dx,$$

where $\beta = (\mu_1 - \alpha)/(2\mu_1)$. Now by Sobolev inequality and Jensen inequality

$$J(w) \geq \beta \int_\Omega |\nabla w|^2 \, dx - C\left(\int_\Omega |\nabla w|^2 \, dx\right)^{(p+1)/2} = \beta \|w\|_2^2 - C\|w\|_2^{p+1}.$$

Finally, choosing $\rho := \|w\|_2$ so small such that $0 < \alpha := (\beta\rho^2)/2 \leq J(w)$, we complete the proof of $(ii)$.

To show $(iii)$, we use (5.3) and (5.4) to give an upper estimate to $J$ as follows.

$$\begin{aligned}
J(w) \leq& \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx - C\int_\Omega |w + A|^\mu \, dx + \int_\Omega C - G(x, A) \, dx \\
\leq& \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx - C\int_\Omega |w|^\mu \, dx + C.
\end{aligned}$$

Now fix any $w \in X \setminus \{0\}$ and $t > 0$,

$$J(tw) \leq \left(\frac{1}{2}\int_\Omega |\nabla w|^2 \, dx\right) t^2 - \left(C\int_\Omega |w|^\mu \, dx\right) t^\mu + C.$$

Then we may choose $t$ large enough such that $J(tw) < 0$. Setting $w_0 := tw$, $(iii)$ holds. ☐

**Proposition 5.2.3.** *$J$ satisfies the Palais-Smale condition, that is, whenever $\{u_j\}_{j=1}^{\infty} \subset X$ is a sequence with $J(u_j)$ is bounded and $J'(u_j) \to 0$, $\{u_j\}$ contains a convergent subsequence in $X$.*

*Proof.* We first show that $\{u_j\}_{j=1}^{\infty}$ is a bound sequence in $X$. Let $M > 0$ be an upper bound of $J(u_j)$. Notice that as $J'(u_j) \to 0$, we have $|J'(u_j)u_j| \leq \|u_j\|_X$ for large enough $j$. We obtain from (1.7), (1.8) and (5.3) that

$$\begin{aligned}
& M + \frac{1}{\mu}\|u_j\|_X \geq J(u_j) - \frac{1}{\mu}J'(u_j)u_j \\
\geq & \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_j\|_X^2 + \int_\Omega \left(\frac{1}{\mu}g(x, u_j)u_j - G(x, u_j)\right)\, dx \\
\geq & \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_j\|_X^2 + \int_{\{x\in\Omega : |u_j(x)| < r\}} \left(\frac{1}{\mu}g(x, u_j)u_j - G(x, u_j)\right)\, dx.
\end{aligned} \tag{5.7}$$

By (5.2), we see that $|\mu^{-1}g(x, u_j)u_j - G(x, u_j)|$ is bounded by some constant depending on $r$ and $|\Omega|$. It follows from (5.7) that $\|u_j\|_X$ is uniformly bounded.

Consequently, $\{u_j\}$ has a subsequence, which we still call $\{u_j\}_{j=1}^{\infty}$, satisfying

$$\begin{cases}
u_j \rightharpoonup u & \text{weakly in } X, \\
u_j \to u & \text{strongly in } L^r(\Omega) \text{ for all } r \in [1, 2^*), \\
u_j \to u & \text{almost everywhere in } \Omega.
\end{cases} \tag{5.8}$$

With these convergence, we are going to show that $\|u_j\|_X \to \|u\|_X$, which together with the weak convergence above implies that $u_j \to u$ strongly in $X$, hence verifying the Palais-Smale condition.

To show that $\|u_j\|_X \to \|u\|_X$, first note that

$$\begin{aligned}
& \left|\int_\Omega g(x, u_j + A)u_j\, dx - \int_\Omega g(x, u + A)u\, dx\right| \\
\leq & \int_\Omega |g(x, u_j + A)||u_j - u|\, dx + \left|\int_\Omega (g(x, u_j + A) - g(x, u + A))\, u\, dx\right|.
\end{aligned} \tag{5.9}$$

Note also that (5.2) implies $|g(x,z)| \le C + C|z|^p$. Thus we have

$$|g(x,u_j+A)| \le C + C|u_j+A|^p \le C + C|A|^p + C|u_j|^p, \tag{5.10}$$

which ensures the boundedness of $\|g(x,u_j+A)\|_{(p+1)/p}$. Together with the pointwise convergence in (5.8), we conclude the following weak convergence in $L^{p/(p+1)}(\Omega)$.

$$g(x,u_j+A) \rightharpoonup g(x,u+A),$$

which implies $\int_\Omega (g(x,u_j+A) - g(x,u+A))\, u\, dx = o(1)$, as $u \in L^p(\Omega)$. Consequently, (5.9) reads as

$$\begin{aligned}
&\left|\int_\Omega g(x,u_j+A)u_j\, dx - \int_\Omega g(x,u+A)u\, dx\right| \\
&\le \int_\Omega |g(x,u_j+A)||u_j-u|\, dx + o(1) \\
&\le \int_\Omega \left(|g(x,0)||u_j-u| + \lambda|u_j+A||u_j-u| + \frac{C_1}{p}|u_j+A|^p|u_j-u|\right)\, dx + o(1) \\
&\le \left(\|g(x,0)\|_2 + \lambda\|u_j+A\|_2\right)\|u_j-u\|_2 + \frac{C_1}{p}\|u_j+A\|_{p+1}^{p+1}\cdot\|u_j-u\|_{p+1} + o(1)
\end{aligned}$$

Noting from (5.8) that $u_j \to u$ strongly in $L^2(\Omega)$ and $L^{p+1}(\Omega)$, as $2 < p+1 < 2^*$, we conclude that

$$\begin{aligned}
\int_\Omega g(x,u_j+A)u\, dx &= \int_\Omega g(x,u+A)u\, dx + o(1) \\
\int_\Omega g(x,u_j+A)u_j\, dx &= \int_\Omega g(x,u+A)u\, dx + o(1),
\end{aligned} \tag{5.11}$$

as $j \to \infty$.

On the other hand, we put $u = u_j$ and $\varphi = u$ in (1.8) to get

$$\int_\Omega \nabla u_j \cdot \nabla u\, dx - \int_\Omega g(x,u_j+A)u\, dx = J'(u_j)u = o(1).$$

With the aid of (5.11), we let $j \to \infty$ in the above and obtain

$$\int_\Omega |\nabla u|^2\, dx = \int_\Omega g(x,u+A)u\, dx. \tag{5.12}$$

Similarly, we put $u = u_j$ and $\varphi = u_j$ into (1.8) and utilize the fact that $\|u_j\|_X$ is uniformly bounded to get

$$\int_\Omega \nabla u_j \cdot \nabla u_j\, dx - \int_\Omega g(x,u_j+A)u_j\, dx = J'(u_j)u_j = o(1). \tag{5.13}$$

Then (5.11), (5.12) and (5.13) imply

$$\|u_j\|_X = \|u\|_X + o(1),$$

as $j \to \infty$. □

**Remark 5.2.4.** In the proof of Proposition 5.2.3, we did not utilize the full power of (5.2), but only the following growth control of $g$.

$$|g(x,z)| \leq C + C|z|^p. \tag{5.14}$$

This guarantees that $J$ still satisfies the Palais-Smale condition under a weaker growth condition (5.14), instead of (5.2).

In view of Proposition 5.2.2 and Proposition 5.2.3, the mountain pass lemma applies to conclude that the energy functional $J$ has a nontrivial critical point which is a nontrivial classical solution. We have finished the proof of Theorem 5.0.1.

## 5.3 The constant $A_0$

In this section, we investigate the constant $A_0$ via studying the following special case of Problem (5.1).

$$\begin{cases} -\Delta u = |u|^{p-1}u - \fint_\Omega |u|^{p-1}u \, dx & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \fint_\Omega u \, dx = A, \end{cases} \tag{5.15}$$

where $p$ and $A$ are as before. We see that in the proof of Theorem 5.0.1, the constant $A_0$ is determined when we try to verify condition $(ii)$ of Theorem 5.1.2. We find a suitable $A_0$ explicitly in this case.

**Proposition 5.3.1.** *Let $A_0$ be a constant satisfying $p|A_0|^{p-1} = \mu_1$, where $\mu_1$ is the first eigenvalue of the eigenvalue problem (B.1). Then for each $A \in (-A_0, A_0)$, Problem (5.15) has a nontrivial classical solution.*

*Proof.* As $A \in (-A_0, A_0)$, we have $p|A|^{p-1} < \mu_1$. Let $\tau$ be in $(p|A|^{p-1}, \mu_1)$. Note that the energy associated to Problem (5.15) is

$$J(w) = \frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \frac{1}{p+1}\int_\Omega \left(|w+A|^{p+1} - |A|^{p+1}\right)\,dx. \tag{5.16}$$

To verify condition (*ii*) of Theorem 5.1.2, we estimate the second term on two special subsets of $\Omega$, namely, $\Omega_1 := \{x : |u(x)| \le \sigma|A|\}$ and $\Omega_2 := \{x : |u(x)| > \sigma|A|\}$, where $\sigma$ is a positive real number. By mean value theorem,

$$|w+A|^{p+1} - |A|^{p+1} = (p+1)|A|^{p-1}Aw + \frac{1}{2}\int_0^1 p(p+1)|\theta w + A|^{p-1}w^2\,d\theta.$$

Thus we have

$$\begin{aligned}
&\frac{1}{p+1}\int_{\Omega_1}\left(|w+A|^{p+1} - |A|^{p+1}\right)\,dx - |A|^{p-1}A\int_{\Omega_1} w\,dx \\
&\quad= \frac{p}{2}\int_{\Omega_1}\left(\int_0^1 |\theta w + A|^{p-1}w^2\,d\theta\right)dx \\
&\quad\le \frac{p}{2}\int_{\Omega_1}(|w| + |A|)^{p-1}\,w^2\,dx \\
&\quad\le \frac{p}{2}(1+\sigma)^{p-1}|A|^{p-1}\int_\Omega w^2\,dx.
\end{aligned}$$

Choosing $\sigma > 0$ so small that $p|A|^{p-1}(1+\sigma)^{p-1} < \tau$, we get the estimate on $\Omega_1$

$$\frac{1}{p+1}\int_{\Omega_1}\left(|w+A|^{p+1} - |A|^{p+1}\right)\,dx - |A|^{p-1}A\int_{\Omega_1} w\,dx < \frac{\tau}{2}\int_\Omega w^2\,dx. \tag{5.17}$$

Similarly, we have

$$\begin{aligned}
&\frac{1}{p+1}\int_{\Omega_2}\left(|w+A|^{p+1} - |A|^{p+1}\right)\,dx - |A|^{p-1}A\int_{\Omega_2} w\,dx \\
&\quad\le \frac{p}{2}\int_{\Omega_2}(|w| + |A|)^{p-1}\,w^2\,dx \\
&\quad\le C\int_{\Omega_2}\left(|w|^{p+1} + |A|^{p-1}w^2\right)\,dx \qquad (5.18)\\
&\quad\le C\|\nabla w\|_2^{p+1} + C|A|^{p-1}\|w\|_{2^*}^2\left(\int_{\Omega_2} 1\right)^{(2^*-2)/2^*} \\
&\quad\le C\|\nabla w\|_2^{p+1} + C|A|^{p-1}|\Omega_2|^{2/n}\|\nabla w\|_2^2.
\end{aligned}$$

To estimate $|\Omega_2|$ we have,

$$\sigma|A||\Omega_2| = \sigma|A| \int_{\Omega_2} 1 \; dx \leq \int_{\Omega_2} |w| \; dx \leq C\|\nabla w\|_2.$$

Without loss of generality we may assume $A \neq 0$ and hence

$$|\Omega_2|^{2/n} \leq C \left(\frac{1}{\sigma|A|}\right)^{2/n} \|\nabla w\|_2^{2/n}.$$

Putting it in (5.18) to get

$$\begin{aligned} \frac{1}{p+1} \int_{\Omega_2} \left(|w+A|^{p+1} - |A|^{p+1}\right) \; dx - |A|^{p-1}A \int_{\Omega_1} w \; dx \\ \leq C\|\nabla w\|_2^{p+1} + C|A|^{p-1} \left(\frac{1}{\sigma|A|}\right)^{2/n} \|\nabla w\|_2^{2+(2/n)}. \end{aligned} \tag{5.19}$$

Combining (5.17) and (5.19), we get

$$\begin{aligned} &\frac{1}{p+1} \int_{\Omega} \left(|w+A|^{p+1} - |A|^{p+1}\right) \; dx \\ &\leq (1+\sigma)^{p-1} \frac{p}{2} |A|^{p-1} \int_{\Omega} w^2 \; dx + C\|\nabla w\|_2^{p+1} + \\ &C|A|^{p-1} \left(\frac{1}{\sigma|A|}\right)^{2/n} \|\nabla w\|_2^{2+(2/n)}. \end{aligned} \tag{5.20}$$

Together with (5.16), we have

$$\begin{aligned} J(w) \geq& \frac{1}{2}\left(\|\nabla w\|_2^2 - \tau \int_{\Omega} w^2 \; dx\right) - C\|\nabla w\|_2^{p+1} \\ &- C|A|^{p-1} \left(\frac{1}{\sigma|A|}\right)^{2/n} \|\nabla w\|_2^{2+(2/n)} \\ \geq& \beta\|\nabla w\|_2^2 - C\|\nabla w\|_2^{p+1} - C|A|^{p-1} \left(\frac{1}{\sigma|A|}\right)^{2/n} \|\nabla w\|_2^{2+(2/n)}, \end{aligned}$$

for some $\beta > 0$. We finally choose $\rho = \|\nabla w\|_2$ so small that

$$J(w) \geq \frac{\beta}{2}\rho^2 =: \alpha > 0,$$

and complete the verification of $(ii)$. □

# Chapter 6

# The Superlinear-Subcritical Case III

In this chapter we deal with the following problem

$$\begin{cases} -\Delta u = g(x,u) - \fint_\Omega g(x,u)\, dx & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \fint_\Omega u\, dx = 0, \end{cases} \tag{6.1}$$

where $g(x,z)$ is in $C^1(\overline{\Omega} \times \mathbb{R})$ and odd in $z$. We can see that the corresponding energy $J$ reads as

$$J(w) = \frac{1}{2}\int_\Omega |\nabla w|^2\, dx - \int_\Omega G(x,w)\, dx \tag{6.2}$$

for $w \in X$, and hence $J$ is an even functional. Let $p$ be in $(1,(n+2)/(n-2))$. We impose the following growth controls on $g$ which is a weaker growth control comparing to (5.2):

(i) There exists $C > 0$ such that for every $x \in \overline{\Omega}$ and $z \in \mathbb{R}$, we have

$$|g(x,z)| \leq C + C|z|^p. \tag{6.3}$$

(ii) There exist $r > 0$ and $\mu > 2$ such that for every $x \in \overline{\Omega}$ and $z$ with $|z| \geq r$, we have

$$0 < \mu G(x,z) \leq g(x,z)z. \tag{6.4}$$

Thus by integration we again have

$$G(x,z) \geq C|z|^{\mu} - C. \tag{6.5}$$

Here and after in this chapter, the positive constant $C$ will be repeatedly used as the a generic constant which depends only on $n$, $p$ and $\Omega$. We are going to prove the following multiplicity theorem.

**Theorem 6.0.1.** *Problem* (6.1) *processes infinitely many solutions.*

In view of the Theorem 2.1.5, we know that every critical point of $J$ is a classical solution of Problem (6.1). Thus it suffices to show the existence of infinitely many critical points.

## 6.1 A multiplicity theorem

In this section, we state the following multiplicity theorem taken from [12].

**Theorem 6.1.1.** *Let $X$ be an infinitely dimensional Banach space, which can be written as $X = V \oplus W$, where $V$ is a finite dimensional subspace of $X$. Let $J$ be an even $C^1$-functional on $X$ satisfying the Palais-Smale condition with $J(0) = 0$. Suppose further that $J$ has the following mountain pass type behavior near* 0*:*

*(i) There are positive constants $\rho$ and $\alpha$ such that $J|_{\partial B_\rho \cap W} \geq \alpha$, and*

*(ii) For each finite dimensional subspace $Y \subset X$, there is an $R = R(Y)$ such that $J|_{Y \setminus B_R} \leq 0$,*

*then $J$ possesses an unbounded sequence of critical values.*

## 6.2 Existence of infinitely many critical points

In this section, we show the following proposition by Theorem 6.1.1, so to complete the proof of Theorem 6.0.1.

**Proposition 6.2.1.** *The energy functional $J$ processes infinitely many critical points.*

*Proof.* By the study of the eigenvalue problem $(B.1)$, we have an orthonormal basis of $L^2(\Omega)$ say $\{\varphi_j\}_{j=1}^{\infty}$, where $\varphi_j$ is a eigenfunction corresponds to the eigenvalue $\mu_j$. Let $k$ be a natural number. Then we may set $V = V(k) := \{\varphi_1, \varphi_2, ..., \varphi_k\}$ and $W = W(k) := \{\varphi_{k+1}, \varphi_{k+2}, ...\}$. We thus have $W = V^{\perp}$ and hence $X = V \oplus W$. On the other hand, $J(0) = 0$ and $J$ satisfies the Palais-Smale condition by Proposition 5.2.3 and its remark.

By Theorem 6.1.1, in order to show the existence of infinitely many critical points, it suffices to show that $J$ satisfies condition $(i)$ and $(ii)$ of Theorem 6.1.1.

We first show that $J$ satisfies condition $(i)$, by choosing suitable $k$ above. Notice that the growth control (6.3) of $g$, implies the following growth control of $G$ on $\overline{\Omega} \times \mathbb{R}$.

$$|G(x,z)| \leq C + C|z|^{p+1}, \tag{6.6}$$

Thus we obtain a lower estimate for $J$ by (6.2) and (6.6), namely, for every $w \in X$, we have

$$J(w) \geq \frac{1}{2}\int_{\Omega} |\nabla w|^2 \, dx - C\int_{\Omega} |w|^{p+1} \, dx - C. \tag{6.7}$$

Due to the negative constant term $-C$ on the right hand side, we cannot simply apply the Sobolev inequality and choose $\|\nabla w\|_2$ small enough as what we have done in the proof of Proposition 5.2.2. In this case, instead of getting the nonnegativity of $J$ on X, we try to obtain it on the subspace $W = W(k)$.

Consider interpolation inequality and Sobolev inequality, for every $w \in X$ we have

$$\|w\|_{p+1} \leq \|w\|_{2^*}^{\gamma} \cdot \|w\|_2^{1-\gamma} \leq C\|\nabla w\|_2^{\gamma} \cdot \|w\|_2^{1-\gamma}, \tag{6.8}$$

where $\gamma \in (0,1)$ is given by

$$\frac{1}{p+1} = \frac{\gamma}{2^*} + \frac{1-\gamma}{2}.$$

In particular, for every $w \in W = W(k) = V(k)^{\perp} = \operatorname{span}\{\varphi_1, \varphi_2, ..., \varphi_k\}^{\perp}$ we have

$$\|w\|_2^2 = \int_\Omega w^2 \, dx \le \frac{1}{\mu_k} \int_\Omega |\nabla w|^2 \, dx = \|\nabla w\|_2^2. \tag{6.9}$$

Combining (6.8) and (6.9), we obtain

$$\|w\|_{p+1} \le \frac{C}{\mu_k^{(1-\gamma)/2}} \|\nabla w\|_2.$$

Together with (6.7), we have

$$J|_{\partial B_\rho \cap W} \ge \left( \frac{1}{2} - \frac{C}{\mu_k^{(1-\gamma)(p+1)/2}} \rho^{p-1} \right) \rho^2 - C,$$

for any $k \in \mathbb{N}$ and $\rho > 0$. Now for each $\rho > 0$ we can choose $k = k(\rho)$ large enough so that

$$J|_{\partial B_\rho \cap W} \ge \frac{1}{4} \rho^2 - C.$$

We finally choose $\rho > 0$ so that

$$J|_{\partial B_\rho \cap W} \ge \frac{\rho^2}{8} =: \alpha > 0.$$

Thus Condition $(i)$ has been verified.

Now we turn to Condition $(ii)$. Let $Y$ be a finite dimensional subspace of $X$. By (6.5) and (6.2), we have the following upper estimate for $w \in X$.

$$J(w) \le \frac{1}{2} \int_\Omega |\nabla w|^2 \, dx - C \int_\Omega |w|^\mu \, dx + C, \tag{6.10}$$

where $\mu$ is some constant greater than 2. Notice that $w \mapsto \|w\|_\mu^\mu$ is a continuous map on $X$, and that $\partial B_1 \cap Y$ is a compact set, we can hence find a $w_0 \in \partial B_1 \cap Y$ so that

$$\min_{w \in \partial B_1 \cap Y} \left( \int_\Omega |w|^\mu \, dx \right) = \int_\Omega |w_0|^\mu \, dx > 0.$$

which implies for every $t > 0$,

$$\min_{w \in \partial B_t \cap Y} \left( \int_\Omega |w|^\mu \, dx \right) = t^\mu \int_\Omega |w_0|^\mu \, dx > 0.$$

Thus for $w \in \partial B_t \cap Y$, (6.10) reads as

$$J(w) \leq \frac{1}{2} t^2 - \left( \int_\Omega |w_0|^\mu \, dx \right) t^\mu + C. \tag{6.11}$$

Note that the choice of $w_0$ depends on the subspace $Y$ of $X$, thus by (6.11) we can choose a constant $R = R(Y)$ large enough so that

$$J|_{Y \setminus B_R} \leq - \left( \frac{1}{2} \int_\Omega |w_0|^\mu \, dx \right) R^\mu + C < 0.$$

We have verified Condition (*ii*). The desired result follows from Theorem 6.1.1.

$\square$

# Chapter 7

## The Critical Case

In this chapter we deal with the following problem

$$\begin{cases} -\Delta u = |u|^{p-1}u + f(u) - \fint_\Omega \left(|u|^{p-1}u + f(u)\right)\, dx & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \fint_\Omega u\, dx = A, \end{cases} \tag{7.1}$$

where $A$ is a real number, $p = (n+2)/(n-2)$ and $f(z) = \lambda z + \mu|z|^{q-1}z$, for some $q \in (1,p)$, real constant $\lambda$ and nonnegative constant $\mu$. Motivated by [2] and [13], we prove the following theorem concerning the existence of a nontrivial solution to Problem (7.1).

**Theorem 7.0.1.** *(i) Suppose $n \geq 5$, $\lambda$, $\mu$ and $A$ are such that $p|A|^{p-1} + \lambda + \mu q|A|^{q-1} < \mu_1$. Then Problem* (7.1) *has a nontrivial solution.*

*(ii) Suppose $n = 4$. Then there are positive constants $\lambda_0$ and $A_1$ such that for each $\lambda \in (-\infty, \lambda_0)$ and $A \in (-A_1, A_1)$, Problem (7.1) has a nontrivial solution.*

By Theorem 2.1.5, every critical point of $J$ is a classical solution of Problem (1.5). Consequently it suffices to show the existence of a nontrivial critical point.

In this chapter, we use the positive number $C$ to denote a generic constant depending only on $n$, $\Omega$, $q$ and $A$. We further use the subscript $\varepsilon$ (or $\delta$) to emphasize the dependence of the constant on a certain parameter $\varepsilon$ (or $\delta$).

## 7.1 A variant of the mountain pass lemma

Due to the lack of compactness, we cannot apply the standard version of the mountain pass lemma (Theorem 5.1.2) to conclude the existence of a critical point to the energy $J$. In this case, we are going to employ the following variant of the mountain pass lemma taken from [1], to get an approximating sequence converging to some $u$. After that, we will show that this $u$ is indeed a critical point.

**Theorem 7.1.1.** *Let $X$ be a real Banach space and $J$ be a $C^1$-functional on $X$ satisfying the mountain pass property near the origin (see Definition 5.1.1). Setting*

$$c = \inf_{\gamma \in \Gamma} \max_{w \in \gamma} J(w),$$

*where $\Gamma$ is the class of all continuous paths from $0$ to $w_0$. Then there is a sequence $\{u_j\}_{j=1}^{\infty} \subset X$ such that*

$$J(u_j) \to c, \tag{7.2}$$

*and*

$$J'(u_j) \to 0 \text{ in } X^*, \tag{7.3}$$

*as $j \to \infty$.*

## 7.2 An approximating sequence

To employ Theorem 7.1.1, we check the mountain pass property for $J$ near the origin.

**Proposition 7.2.1.** *Suppose $\lambda$, $\mu$ and $A$ are such that $p|A|^{p-1}+\lambda+\mu q|A|^{q-1} < \mu_1$. Then $J$ satisfies the mountain pass property near the origin, that is*

*(i)* $J(0) = 0$,

*(ii) There are positive constants $\rho$ and $\alpha$ such that $J|_{\partial B_\rho(0)} \geq \alpha$, where $\partial B_\rho(0)$ is the set $\{w \in X : \|w\|_X = \rho\}$, and*

*(iii) There exists some $w_0$ in $X$ with $\|w_0\|_X > \rho$ such that $J(w_0) \leq 0$.*

*Proof.* The proof of $(i)$ and $(iii)$ are similar to that of Proposition 5.2.2 and we shall omit them here. We now show that how the estimate $p|A|^{p-1}+\lambda+\mu q|A|^{q-1} < \mu_1$ guarantees $(ii)$.

Let $\nu$ be such that $p|A|^{p-1} + \lambda + \mu q|A|^{q-1} < \nu < \mu_1$ and $\sigma > 0$ be so small that $p|A|^{p-1}(1+\sigma)^{p-1} + \lambda + \mu q|A|^{q-1}(1+\sigma)^{p-1} < \nu$. By a similar argument to the derivation of (5.20) in Proposition 5.3.1, we have

$$\begin{aligned}\frac{1}{p+1}\int_\Omega &\left(|w+A|^{p+1} - |A|^{p+1}\right)\,dx\\ &\leq (1+\sigma)^{p-1}\frac{p}{2}|A|^{p-1}\int_\Omega w^2\,dx + C\|\nabla w\|_2^{p+1} +\\ &\quad C|A|^{p-1}\left(\frac{1}{\sigma|A|}\right)^{2/n}\|\nabla w\|_2^{2+(2/n)},\end{aligned} \tag{7.4}$$

and

$$\begin{aligned}\frac{1}{q+1}\int_\Omega &\left(|w+A|^{q+1} - |A|^{q+1}\right)\,dx\\ &\leq (1+\sigma)^{q-1}\frac{q}{2}|A|^{q-1}\int_\Omega w^2\,dx + C\|\nabla w\|_2^{q+1} +\\ &\quad C|A|^{q-1}\left(\frac{1}{\sigma|A|}\right)^{2/n}\|\nabla w\|_2^{2+(2/n)}.\end{aligned} \tag{7.5}$$

On the other hand, we have

$$\frac{\lambda}{2}\int_\Omega \left(|w+A|^2 - |A|^2\right)\,dx = \frac{\lambda}{2}\int_\Omega w(w+2A)\,dx = \frac{\lambda}{2}\int_\Omega w^2\,dx. \tag{7.6}$$

Combining (7.4), (7.5) and (7.6) into (1.7), we get

$$\begin{aligned}J(w) \geq & \frac{1}{2}\left(\|\nabla w\|_2^2 - \nu \int_\Omega w^2\,dx\right) - C\|\nabla w\|_2^{p+1} - C\|\nabla w\|_2^{q+1} \\ & - C\left(|A|^{p-1} + |A|^{q-1}\right)\left(\frac{1}{\sigma|A|}\right)^{2/n} \|\nabla w\|_2^{2+2/n} \\ \geq & \beta\|\nabla w\|_2^2 - C\|\nabla w\|_2^{p+1} - C\|\nabla w\|_2^{q+1} \\ & - C\left(|A|^{p-1} + |A|^{q-1}\right)\left(\frac{1}{\sigma|A|}\right)^{2/n} \|\nabla w\|_2^{2+2/n},\end{aligned}$$

for some $\beta > 0$. We then choose $\rho = \|w\|_X = \|\nabla w\|_2$ so small that

$$J(w) \geq \frac{\beta}{2}\rho^2 =: \alpha > 0,$$

and complete the proof. □

With this proposition, we apply Theorem 7.1.1 to get a sequence $\{u_j\}_{j=1}^\infty \subset X$ such that (7.2) and (7.3) hold as $j \to \infty$. Next, we derive some convergence about $\{u_j\}$ by showing that $\|u_j\|_X$ is uniformly bounded. Firstly, we rewrite (7.2) and (7.3) as

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla u_j|^2\,dx - \frac{1}{p+1}\int_\Omega \left(|u_j + A|^{p+1} - |A|^{p+1}\right)\,dx \\ - \int_\Omega \left(F(u_j + A) - F(A)\right)\,dx = c + o(1),\end{aligned} \tag{7.7}$$

and

$$\begin{aligned}\int_\Omega \nabla u_j \cdot \nabla\varphi\,dx - \int_\Omega |u_j + A|^{p-1}(u_j + A)\varphi\,dx \\ - \int_\Omega f(u_j + A)\varphi\,dx = \langle \xi_j, \varphi\rangle,\end{aligned} \tag{7.8}$$

for $\varphi$ in $X$, respectively. Note that $\|\xi_j\|_{X^*} \to 0$ as $j \to \infty$. Putting $\varphi = u_j$ in (7.8), we get

$$\int_\Omega |\nabla u_j|^2\,dx - \int_\Omega |u_j + A|^{p-1}(u_j + A)u_j\,dx - \int_\Omega f(u_j + A)u_j\,dx = \langle \xi, u_j\rangle,$$

and hence

$$\begin{aligned}\int_\Omega |\nabla u_j|^2\,dx - \int_\Omega \left(|u_j + A|^{p+1} - |A|^{p+1}\right)\,dx - \int_\Omega f(u_j + A)u_j\,dx \\ +A\int_\Omega \left(|u_j + A|^{p-1}(u_j + A) - A|A|^{p-1}\right)\,dx = \langle \xi_j, u_j\rangle.\end{aligned} \tag{7.9}$$

Subtract half of (7.9) from (7.7) to get

$$
\begin{aligned}
&\frac{1}{n}\int_\Omega \left(|u_j+A|^{p+1}-|A|^{p+1}\right)\,dx\\
=&\frac{A}{2}\int_\Omega \left((u_j+A)|u_j+A|^{p-1}-A|A|^{p-1}\right)\,dx+\int_\Omega (F(u_j+A)-F(A))\,dx\\
&-\frac{1}{2}\int_\Omega f(u_j+A)u_j\,dx+c+\langle -\xi_j,u_j\rangle+o(1).
\end{aligned}
\tag{7.10}
$$

We then obtain the estimate

$$
\begin{aligned}
\frac{1}{n}\int_\Omega |u_j|^{p+1}\,dx\le\frac{|A|}{2}\int_\Omega |u_j+A|^p\,dx&+\int_\Omega (F(u_j+A)-F(A))\,dx\\
&+\frac{1}{2}\int_\Omega |f(u_j+A)u_j|\,dx+\langle -\xi_j,u_j\rangle+c+o(1).
\end{aligned}
\tag{7.11}
$$

We have to estimate the first three terms on the right hand side of (7.11) one by one. Note that for every positive $\varepsilon$, we have $|z|^p$, $|z|^q$, $|z|^2$, $|z|\le\varepsilon|z|^{p+1}+C_\varepsilon$ on $\mathbb{R}$. Thus we obtain an estimate for the first term on the right hand side of (7.11),

$$
\begin{aligned}
\int_\Omega |u_j+A|^p\,dx\le&C\int_\Omega (|u_j|^p+|A|^p)\,dx\\
\le&\varepsilon C\int_\Omega |u_j|^{p+1}\,dx+C_\varepsilon.
\end{aligned}
\tag{7.12}
$$

For the second term, we estimate as follows:

$$
\begin{aligned}
\int_\Omega &(F(u_j+A)-F(A))\,dx=\int_\Omega f(\theta u_j+A)u_j\,dx\\
&\le\int_\Omega \left(\varepsilon|\theta u_j+A|^p+C_\varepsilon|\theta u_j+A|\right)|u_j|\,dx\\
&\le\int_\Omega \left(\varepsilon C(|u_j|^p+|A|^p)+C_\varepsilon(|u_j|+|A|)\right)|u_j|\,dx\\
&=\varepsilon C\int_\Omega |u_j|^{p+1}\,dx+C_\varepsilon\int_\Omega |u_j|^2\,dx+C_\varepsilon\int_\Omega |u_j|\,dx\\
&\le\varepsilon C\int_\Omega |u_j|^{p+1}\,dx+\delta C_\varepsilon\int_\Omega |u_j|^{p+1}\,dx+C_\varepsilon C_\delta.
\end{aligned}
$$

In other words, we obtain the following estimate for the second term on the right hand side of (7.11),

$$
\int_\Omega (F(u_j+A)-F(A))\,dx\le(\varepsilon C+\delta C_\varepsilon)\int_\Omega |u_j|^{p+1}\,dx+C_\varepsilon C_\delta.
\tag{7.13}
$$

Using a similar argument, we obtain an estimate for the third term,

$$\int_\Omega |f(u_j + A)u_j| \; dx \le (\varepsilon C + \delta C_\varepsilon) \int_\Omega |u_j|^{p+1} \; dx + C_\varepsilon C_\delta. \tag{7.14}$$

Combining the estimates (7.12), (7.13), (7.14) into (7.11) we get,

$$\begin{aligned}\frac{1}{n} \int_\Omega & |u_j + A|^{p+1} \; dx \\ &\le (\varepsilon C + \delta C_\varepsilon) \int_\Omega |u_j|^{p+1} \; dx + \|\xi_j\|_{X^*} \|u_j\|_X + C_\varepsilon C_\delta + c + o(1).\end{aligned}$$

Now utilize

$$\int_\Omega |u_j + A|^{p+1} \; dx \ge 2^{-p} \int_\Omega |u_j|^{p+1} \; dx - \int_\Omega |A|^{p+1} \; dx$$

and choose $\varepsilon$ then $\delta$ small enough to get

$$\frac{1}{2n} \int_\Omega |u_j|^{p+1} \; dx \le \|\xi_j\|_{X^*} \|u_j\|_X + C.$$

As $\|\xi_j\|_{X^*} = o(1)$, we have

$$\int_\Omega |u_j|^{p+1} \; dx \le \|u_j\|_X + C, \tag{7.15}$$

and

$$\frac{1}{p+1} \int_\Omega \left( |u_j + A|^{p+1} - |A|^{p+1} \right) \; dx \le C + C\|u_j\|_X, \tag{7.16}$$

for sufficiently large $j$. Now combine (7.16) with (7.7) to get

$$\frac{1}{2}\|u_j\|_X^2 \le C + C\|u_j\|_X + \int_\Omega (F(u_j + A) - F(A)) \; dx + c + 1. \tag{7.17}$$

By (7.13), (7.15) and (7.17), we eventually conclude that $\|u_j\|_X$ is bounded by some positive constant $M$.

Consequently $\{u_j\}_{j=1}^\infty$ contains a subsequence, which we still call $\{u_j\}$, converging to some $u \in X$ satisfying

$$\begin{cases} u_j \rightharpoonup u & \text{weakly in } X, \\ u_j \to u & \text{strongly in } L^r \text{ for } 1 \le r \le p+1, \\ u_j \to u & \text{almost everywhere in } \Omega, \\ (u_j + A)|u_j + A|^{p-1} \rightharpoonup (u + A)|u + A|^{p-1} & \text{weakly in } \left(L^{p+1}\right)^*, \\ f(u_j + A) \rightharpoonup f(u + A) & \text{weakly in } \left(L^{p+1}\right)^*. \end{cases} \tag{7.18}$$

On the other hand, from the boundedness of $\|u_j\|_X$ we have $|\langle \xi_j, u_j \rangle| \leq M\|\xi_j\|_{X^*} \to 0$. Thus (7.8) becomes

$$\int_\Omega \nabla u_j \cdot \nabla \varphi \, dx - \int_\Omega (u_j + A)|u_j + A|^{p-1} \varphi \, dx - \int_\Omega f(u_j + A)\varphi \, dx = o(1)$$

for any fixed $\varphi \in X$. Utilizing the weak convergence in (7.18), we obtain

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx - \int_\Omega (u + A)|u + A|^{p-1} \varphi \, dx - \int_\Omega f(u + A)\varphi = 0, \qquad (7.19)$$

for all $\varphi \in X$. This proves that the limiting function $u$ is a critical point of the energy functional $J$.

We summarize this section with the following proposition, which we have just proven in the above discussion.

**Proposition 7.2.2.** *With $\lambda$, $\mu$ and $A$ as in Proposition* 7.2.1, *there exist a sequence $\{u_j\}_{j=1}^\infty \subset X$ converging to some $u \in X$, in the sense of (7.18). Moreover, $u$ is a critical point of the energy functional $J$.*

**Remark 7.2.3.** Note that in this case $p = (n+2)/(n-2) = 2^* - 1$, we do not have the convergence $\|u_j - u\|_{p+1} \to 0$ and hence the key argument in Proposition 5.2.3, namely, (5.11) does not work here. This means that the minimax value $c$ may not be attained by some critical point. This forces us to the following discussion.

## 7.3 A sufficient condition for a critical point to be nontrivial

To show the existence of a nontrivial critical point, we first derive a sufficient condition for the critical point $u$ above to be nontrivial. In the next section, we will show how the sufficient condition is satisfied.

**Proposition 7.3.1.** *Let $\lambda$, $\mu$ and $A$ be as in Proposition 7.2.1. Let $c$ be the minimax value given by Theorem 7.1.1. Let $S := \inf_{u\in H_0^1(\Omega)} \left(\|\nabla u\|_2^2/\|u\|_{p+1}^2\right)$ be the best constant of the Sobolev inequality. Suppose $c < (2n)^{-1}S^{n/2}$. Then $u$ is nontrivial, that is, $u$ is not identically equal to zero. Moreover, we have $J(u) \leq c$.*

*Proof of Proposition 7.3.1.* Suppose $u$ is identically equal to zero. By the convergence in (7.18), we have

$$\int_\Omega (u_j + A)|u_j + A|^{p-1} - A|A|^{p-1}\, dx \to \int_\Omega (0 + A)|0 + A|^{p-1} - A|A|^{p-1} = 0,$$

and $\langle u_j, \xi_j\rangle \to 0$. Thus (7.10) becomes

$$\begin{aligned} &\frac{1}{n}\int_\Omega \left(|u_j + A|^{p+1} - |A|^{p+1}\right)\, dx \\ &\qquad = \int_\Omega (F(u_j + A) - F(A))\, dx - \frac{1}{2}\int_\Omega f(u_j + A)u_j\, dx + c + o(1). \end{aligned} \tag{7.20}$$

Note that we have

**Lemma 7.3.2.** *With $\lambda$, $\mu$, $A$ as in Proposition 7.2.1 and $\{u_j\}$, $u$ as in Proposition 7.2.2, we have*

$$\begin{aligned} &\int_\Omega f(u_j + A)u_j\, dx \to \int_\Omega f(u + A)u\, dx, \quad \text{and} \\ &\int_\Omega (F(u_j + A) - F(A))\, dx \to \int_\Omega (F(u + A) - F(A))\, dx. \end{aligned} \tag{7.21}$$

With this result, (7.20) becomes

$$\int_\Omega \left(|u_j + A|^{p+1} - |A|^{p+1}\right)\, dx = nc + o(1). \tag{7.22}$$

Similarly, (7.9) and (7.22) gives,

$$\int_\Omega |\nabla u_j|^2\, dx = nc + o(1). \tag{7.23}$$

Now for any $\sigma > 0$, we have

$$
\begin{aligned}
&\int_\Omega \left(|u_j + A|^{p+1} - |A|^{p+1}\right)\, dx \\
&= \int_{\{|u_j| \le \sigma |A|\}} \left(|u_j + A|^{p+1} - |A|^{p+1}\right)\, dx + \int_{\{|u_j| > \sigma |A|\}} \left(|u_j + A|^{p+1} - |A|^{p+1}\right) dx \\
&\le \int_{\{|u_j| \le \sigma |A|\}} \left(|u_j + A|^{p+1} - |A|^{p+1}\right)\, dx \\
&\qquad + \int_{\{|u_j| > \sigma |A|\}} \left(1 + \frac{1}{\sigma}\right)^{p+1} |u_j|^{p+1}\, dx - \int_{\{|u_j| > \sigma |A|\}} |A|^{p+1}\, dx \\
&\le \int_{\{|u_j| \le \sigma |A|\}} \left(|u_j + A|^{p+1} - |A|^{p+1}\right)\, dx + \left(1 + \frac{1}{\sigma}\right)^{p+1} \int_\Omega |u_j|^{p+1}\, dx.
\end{aligned}
$$

We then estimate the term $\int_\Omega |u_j|^{p+1}\, dx$ via an inequality taken from [13], namely, there is a positive constant $\varepsilon_0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and $w \in H^1(\Omega)$, we have

$$
\begin{aligned}
&\left(\int_\Omega |w|^{p+1}\, dx\right)^{2/(p+1)} \\
&\quad \le (1+\varepsilon)(2^{-2/n} S - \varepsilon)^{-1} \int_\Omega |\nabla w|^2\, dx + (2^{-2/n} S - \varepsilon)^{-1} C_\varepsilon \int_\Omega |w|^2\, dx.
\end{aligned}
\tag{7.24}
$$

Putting $w = u_j$, we get

$$
\begin{aligned}
&\int_\Omega \left(|u_j + A|^{p+1} - |A|^{p+1}\right)\, dx \\
&\le \int_{\{|u_j| \le \sigma |A|\}} \left(|u_j + A|^{p+1} - |A|^{p+1}\right)\, dx + \left(1 + \frac{1}{\sigma}\right)^{p+1} \\
&\quad \left\{(1+\varepsilon)(2^{-2/n} S - \varepsilon)^{-1} \int_\Omega |\nabla u_j|^2\, dx + (2^{-2/n} S - \varepsilon)^{-1} C_\varepsilon \int_\Omega |u_j|^2\, dx\right\}^{(p+1)/2}.
\end{aligned}
\tag{7.25}
$$

Combine this with (7.22) and (7.23) then let $j \to 0$ to get

$$
nc \le \left(1 + \frac{1}{\sigma}\right)^{p+1} \left\{(1+\varepsilon)(2^{-2/n} S - \varepsilon)^{-1} nc\right\}^{(p+1)/2}.
$$

Notice that the second and the third terms of (7.25) vanish by the dominated convergence theorem and (7.18). Finally, we let $\sigma \to \infty$ to obtain

$$
nc \le \left\{(1+\varepsilon)(2^{-2/n} S - \varepsilon)^{-1} nc\right\}^{(p+1)/2},
$$

which implies that

$$\frac{1}{n}\left(\frac{2^{-2/n}S-\varepsilon}{1+\varepsilon}\right)^{n/2} \le c. \tag{7.26}$$

But it contradicts $c < (2n)^{-1}S^{n/2}$, for sufficiently small $\varepsilon > 0$. This completes the proof of the nontriviality part.

We now show that $J(u) \le c$. By the Lieb's lemma from [10] and convergence in (7.18), we have

$$\begin{aligned}&\int_\Omega \left(|u_j + A|^{p+1} - |A|^{p+1}\right)\, dx \\ &\qquad = \int_\Omega |u_j - u|^{p+1}\, dx + \int_\Omega \left(|u + A|^{p+1} - |A|^{p+1}\right)\, dx + o(1),\end{aligned} \tag{7.27}$$

and

$$\int_\Omega |\nabla u_j|^2\, dx = \int_\Omega |\nabla (u_j - u)|^2\, dx + \int_\Omega |\nabla u|^2\, dx + o(1). \tag{7.28}$$

Then by (7.21) and (7.27), (7.7) reads as

$$\begin{aligned} c =& \frac{1}{2}\int_\Omega |\nabla u_j|^2\, dx - \frac{1}{p+1}\int_\Omega \left(|u_j + A|^{p+1} - |A|^{p+1}\right)\, dx \\ &+ \int_\Omega F(x, u_j + A) - F(x, A)\, dx + o(1) \\ =& J(u) + \left[\frac{1}{2}\int_\Omega |\nabla (u_j - u)|^2\, dx - \frac{1}{p+1}\int_\Omega |u_j - u|^{p+1}\, dx\right] + o(1). \end{aligned} \tag{7.29}$$

On the other hand, as $u$ is a critical point of the energy $J$, we have

$$\int_\Omega |\nabla u|^2\, dx - \int_\Omega |u + A|^{p-1}(u + A)\, dx - \int_\Omega f(u + A)u\, dx = 0.$$

Together with (7.9), (7.21) and (7.28), we get

$$\int_\Omega |\nabla (u_j - u)|^2\, dx - \int_\Omega |u_j - u|^{p+1}\, dx = o(1). \tag{7.30}$$

Finally, we combine (7.29) and (7.30) to get

$$J(u) = c - \frac{1}{n}\int_\Omega |u_j - u|^{p+1}\, dx + o(1),$$

which implies $J(u) \le c$ and we are done. □

In the above proof, we have assumed the validity of lemma 7.3.2 and we prove it now.

*Proof of Lemma 7.3.2.* To verify the first convergence in (7.21), consider

$$\begin{aligned}
&\left|\int_\Omega f(u_j + A)\, dx - \int_\Omega f(u + A)\, dx\right| \\
&\qquad \le \left|\int_\Omega f(u_j + A)u_j\, dx - \int_\Omega f(u_j + A)u\, dx\right| \\
&\qquad\qquad + \left|\int_\Omega f(u_j + A)u\, dx - \int_\Omega f(u + A)u\, dx\right| \\
&\qquad \le \int_\Omega |f(u_j + A)||u_j - u|\, dx + o(1),
\end{aligned}$$

where the last estimate follows from the weak convergence in (7.18). Note that $f(z) = \lambda z + \mu|z|^{q-1}z$, we have the growth control $|f(z)| \le \varepsilon|z|^p + C_\varepsilon|z|$. Thus for each $\varepsilon > 0$ and $j \in \mathbb{N}$, we have

$$\begin{aligned}
&\left|\int_\Omega f(u_j + A)\, dx - \int_\Omega f(u + A)\, dx\right| \\
&\qquad \le \int_\Omega (\varepsilon|u_j + A|^p + C_\varepsilon|u_j + A|)\,|u_j - u|\, dx + o(1) \\
&\qquad \le \varepsilon\|u_j + A\|_{p+1}^p\|u_j - u\|_{p+1} + C_\varepsilon\|u_j + A\|_2\|u_j - u\|_2 + o(1) \\
&\qquad \le \varepsilon(\|u_j\|_{p+1} + \|A\|_{p+1})^p(\|u_j\|_{p+1} + \|u\|_{p+1}) \\
&\qquad\quad + C_\varepsilon(\|u_j\|_2 + \|A\|_2)o(1) + o(1)
\end{aligned}$$

Notice that as $\|u_j\|_X$ is bounded, Sobolev inequality and Jensen inequality yield the boundedness of $\|u_j\|_{p+1}$ and $\|u_j\|_2$. Consequently we have

$$\left|\int_\Omega f(u_j + A)\, dx - \int_\Omega f(u + A)\, dx\right| \le \varepsilon C + C_\varepsilon o(1) + o(1).$$

Taking lim sup for $j \to \infty$ and then $\varepsilon \to 0^+$, we conclude that

$$\left|\int_\Omega f(u_j + A)\, dx - \int_\Omega f(u + A)\, dx\right| \to 0.$$

This completes the proof of the first convergence of (7.21).

To show the second convergence in (7.21), we employ the mean value theorem and use a similar argument as above to obtain

$$
\begin{aligned}
\left|\int_\Omega F(u_j+A)\,dx - \int_\Omega F(u+A)\,dx\right| \\
&= \left|\int_\Omega f(\theta(u_j-u)+u+A)(u_j-u)\,dx\right| \\
&\le \int_\Omega |f(\theta(u_j-u)+u+A)||(u_j-u)|\,dx \\
&\le \varepsilon\int_\Omega (|u_j-u|+|u|+|A|)^p|u_j-u|\,dx \\
&\quad + C_\varepsilon\int_\Omega (|u_j-u|+|u|+|A|)|u_j-u|\,dx \\
&\le \varepsilon\|(|u_j-u|+|u|+|A|)\|_{p+1}^p\|u_j-u\|_{p+1} \\
&\quad + C_\varepsilon\|u_j-u\|_2^2 + C_\varepsilon\|(|u|+|A|)\|_2\|u_j-u\|_2.
\end{aligned}
$$

Similarly, the boundedness of $\|u_j\|_X$ and (7.18) guarantee that

$$\left|\int_\Omega F(u_j+A)\,dx - \int_\Omega F(u+A)\,dx\right| \to 0.$$

This completes the proof. □

## 7.4 A suitable choice of $w_0$

In view of Proposition 7.3.1, the critical point $u$ is nontrivial if $c < (2n)^{-1}S^{n/2}$, where the minimax value $c$ depends on the energy functional $J$ and the choice of $w_0$ given by Proposition 7.2.1. In other words, we have to choose a suitable $w_0$ such that $c < (2n)^{-1}S^{n/2}$ holds.

Notice that in the proof of Proposition 7.2.1, $w_0$ is obtained by scaling an arbitrarily chosen function, say $v \in X$. Together with the fact that $t \mapsto tw_0$ lies in $\Gamma$, we have

$$c := \inf_{\gamma\in\Gamma}\sup_{w\in\gamma} J(w) \le \sup_{0\le t\le 1} J(tw_0) \le \sup_{t\ge 0} J(tw_0) = \sup_{t\ge 0} J(tv). \tag{7.31}$$

The most important implication of (7.31) is the freedom on the right hand side. More precisely, as $v$ is arbitrarily chosen, we can find a suitable choice of $v$ such that $\sup_{t\geq 0} J(tv) < (2n)^{-1}S^{n/2}$ which implies $c < (2n)^{-1}S^{n/2}$.

To find such a suitable candidate $w_0$, we introduce the following family of functions

$$u_\varepsilon := \varepsilon^{(n-2)/4}\frac{1}{(\varepsilon + |x|^2)^{(n-2)/2}}, \tag{7.32}$$

which is the optimal function for Sobolev inequality. However, as it is not in $X$, we consider instead

$$w_\varepsilon := u_\varepsilon - \fint u_\varepsilon \, dx. \tag{7.33}$$

Note that $u_\varepsilon$ and $w_\varepsilon$ are concentrated at the origin. To utilize the concentration behavior, we shall transform the problem back to the origin by imposing some rigid motions on the domain $\Omega$. More precisely, we consider the followings:

Pick $R > 0$ and $x_0 \in \mathbb{R}^n$ such that $\Omega \subset B(x_0, R)$ and $\partial\Omega \cap \partial B(x_0, R) \neq \emptyset$. Let $x_1$ be in $\partial\Omega \cap \partial B(x_0, R)$. As Problem (1.1) is invariance under rigid motions and relabeling of axes, we may without loss of generality assume that $x_1 = 0$ and $\Omega \subset \{x_n > 0\}$. As $\partial\Omega$ is $C^2$, principal curvatures are well defined. Let $\alpha_1, \cdots, \alpha_{n-1}$ be the principal curvatures at 0 with respect to the inner normal. Then there is a small positive constant $\delta$ such that $\partial\Omega$ is the graph of a $C^2$-function near the origin, say, $h : D_\delta \to \mathbb{R}$ with

$$h(x') = \frac{1}{2}\sum_{i=1}^{n-1} \alpha_i x_i^2 + o(|x'|^2) =: g(x') + o(|x'|^2),$$

where $D_\delta = \{x' \in \mathbb{R}^{n-1} : |x'| < \delta\}$ is the $\delta$ disk in $\mathbb{R}^{n-1}$. Hereafter, we automatical impose the above assumption on $\Omega$.

**Proposition 7.4.1.** *Let $\lambda$, $\mu$ and $A$ be as in Proposition* 7.2.1.

*(i) For $n \geq 5$, there is a positive constant $\varepsilon_0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, we have $\sup_{t\geq 0} J(tw_\varepsilon) < (2n)^{-1}S^{n/2}$;*

*(ii) For $n = 4$, there are positive constants $A_0$ and $\varepsilon_0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, if $A \in (-A_0, \infty)$, then we have $\sup_{t\geq 0} J(tw_\varepsilon) < (2n)^{-1}S^{n/2}$.*

**Remark 7.4.2.** (i) In view of this proposition, we simply take $v$ in (7.31) to be $w_\varepsilon$. Then the nontrivial criterion follows from (7.31) and Proposition 7.4.1.

(ii) In view of (*ii*) above, given any $\mu > 0$, we first choose $\lambda_0$ and $A_0'$ so small that $p|A_0'|^{p-1} + \lambda_0 + \mu q|A_0'|^{q-1} < \mu_1$ holds. Then choose $A_1$ in Theorem 7.0.1 to be the minimum of $A_0$ and $A_0'$, so to conclude that for any $\lambda \in (-\infty, \lambda_0)$ and $A \in (-A_1, A_1)$, we have $\sup_{t\geq 0} J(tw_\varepsilon) < (2n)^{-1}S^{n/2}$. This explains how do we choose the constants $\lambda_0$ and $A_1$ in Theorem 7.0.1.

*Proof.* Recall from (1.7) that for $w \in X$

$$\begin{aligned} J(w) =& \frac{1}{2}\int_\Omega |\nabla w|^2\, dx - \frac{1}{p+1}\int_\Omega \left(|w+A|^{p+1} - |A|^{p+1}\right)\, dx \\ & - \int_\Omega (F(w+A) - F(A))\, dx. \end{aligned}$$

Instead of studying the energy $J$ directly, we first study the main term

$$\Phi(w) = \frac{1}{2}\int_\Omega |\nabla w|^2\, dx - \frac{1}{p+1}\int_\Omega |w|^{p+1}\, dx, \tag{7.34}$$

which is closely related to the energy $J$ and has a simpler structure.

Fix $\varepsilon > 0$ and let $T_\varepsilon > 0$ be such that $\Phi(T_\varepsilon w_\varepsilon) = \sup_{t\geq 0}\Phi(tw_\varepsilon)$. Note that the existence of such $T_\varepsilon \geq 0$ is guaranteed by mountain pass property (*ii*), while the positivity follows from $\Phi(0) = 0$ and the proof of the mountain pass property (*ii*). Note also that

$$\Phi(tw_\varepsilon) = \left(\frac{1}{2}\int_\Omega |\nabla w_\varepsilon|^2\, dx\right)t^2 - \left(\frac{1}{p+1}\int_\Omega |w_\varepsilon|^{p+1}\, dx\right)t^{p+1}.$$

By differentiating the above, we have

$$\left(\int_\Omega |\nabla w_\varepsilon|^2\, dx\right)T_\varepsilon - \left(\int_\Omega |w_\varepsilon|^{p+1}\, dx\right)T_\varepsilon^{p+1} = 0. \tag{7.35}$$

This implies

$$T_\varepsilon = \left\{\frac{\displaystyle\int_\Omega |\nabla w|^2\, dx}{\displaystyle\int_\Omega |w_\varepsilon|^{p+1}\, dx}\right\}^{1/(p+1)}, \tag{7.36}$$

hence by Proposition $C.1$, $C.4$ and $C.8$, we have

$$\begin{aligned}\Phi(T_\varepsilon w_\varepsilon) &= \frac{1}{n}\left\{\frac{\displaystyle\int_\Omega |\nabla w_\varepsilon|^2\, dx}{\left(\displaystyle\int_\Omega |w_\varepsilon|^{p+1}\, dx\right)^{(n-2)/n}}\right\}^{n/2} \\ &= \frac{1}{n}\left\{\frac{\frac{1}{2}K_1 - I(\varepsilon) + o(\sqrt{\varepsilon}\,)}{\left[\frac{1}{2}K_4 - II(\varepsilon) + o(\sqrt{\varepsilon}\,)\right]^{(n-2)/n}}\right\}^{n/2} \\ &= \frac{1}{n}\left\{\frac{\frac{1}{2}K_1 - I(\varepsilon)}{\left[\frac{1}{2}K_4 - II(\varepsilon)\right]^{(n-2)/n}}\right\}^{n/2} + o(\sqrt{\varepsilon}\,) \\ &= \frac{1}{n}\left\{2^{-2/n}S - \gamma II(\varepsilon) + o(\sqrt{\varepsilon}\,)\right\}^{n/2} + o(\sqrt{\varepsilon}\,) \\ &= \frac{1}{n}\left\{\left(2^{-2/n}S\right)^{n/2} - \frac{n}{2}(2^{-2/n}S)^{(n-2)/2}\gamma II(\varepsilon)\right\} + o(\sqrt{\varepsilon}\,),\end{aligned}$$

where $K_1$ and $K_4$ are positive constants satisfying

$$S = \frac{K_1}{K_4^{(n-2)/n}},$$

$\gamma$ is a positive constant, $I$ and $II$ satisfy

$$I(\varepsilon) = K_2\sqrt{\varepsilon} + o(\sqrt{\varepsilon})$$

and

$$II(\varepsilon) = K_5\sqrt{\varepsilon} + o(\sqrt{\varepsilon}),$$

for some positive constants $K_2$ and $K_5$. We therefore have

$$\Phi(T_\varepsilon w_\varepsilon) = \frac{1}{2n}S^{n/2} - \gamma' II(\varepsilon) + o(\sqrt{\varepsilon}\,). \tag{7.37}$$

In the following, we will make use of the above study of $\Phi$ to estimate $\sup_{t\geq 0} J(tw_\varepsilon)$ for different $n$, namely $n \geq 6$, $n = 5$, $n = 4$.

**THE CASE $n \geq 6$**

In this case, since $p+1 \in (2,3]$, we thus can utilize the following inequality whose proof is given in appendix A.

$$|x + A|^{p+1} - |A|^{p+1} \geq |x|^{p+1} - \frac{(p+1)p}{2}|A|^{p-1}x^2 + (p+1)A|A|^{p-1}x,$$

for all real numbers $x$ and $A$, with which we are able to compare $J$ and $\Phi$,

$$\begin{aligned}
J(w) \leq& \frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \frac{1}{p+1}\int_\Omega |w|^{p+1}\,dx \\
&+ \frac{p}{2}|A|^{p-1}\int_\Omega |w|^2\,dx - \int_\Omega (F(w+A)-F(A))\,dx \\
=& \Phi(w) + \frac{p}{2}|A|^{p-1}\int_\Omega |w|^2\,dx - \int_\Omega (F(w+A)-F(A))\,dx =: E(w).
\end{aligned}$$

We see that the original energy $J$ is dominated by another energy E.

We further let $t_\varepsilon > 0$ be such that $E(t_\varepsilon w_\varepsilon) = \sup_{t\geq 0} E(tw_\varepsilon)$. Note that the existence of such $t_\varepsilon \geq 0$ is guaranteed by mountain pass property $(i)$, $(ii)$ and the fact that $F(z)$ is a lower order perturbation with respect to $|z|^{p+1}$ as $z \to \infty$. On the other hand, we may without loss of generality assume that $t_\varepsilon > 0$; otherwise we have $\sup_{t\geq 0} E(tw_\varepsilon) = (t_\varepsilon w_\varepsilon) = 0 < (2n)^{-1}S^{n/2}$ and we are done. Again by differentiating, we get

$$\begin{aligned}
&\left(\int_\Omega |\nabla w_\varepsilon|^2\,dx\right)t_\varepsilon - \left(\int_\Omega |w_\varepsilon|^{p+1}\,dx\right)t_\varepsilon^p \\
&\qquad + \left(p|A|^{p-1}\int_\Omega |w_\varepsilon|^2\,dx\right)t_\varepsilon - \int_\Omega f(t_\varepsilon w_\varepsilon + A)w_\varepsilon\,dx = 0,
\end{aligned}$$

which implies

$$\begin{aligned}
\left(\int_\Omega |w_\varepsilon|^{p+1}\,dx\right)t_\varepsilon^p \leq& \left(\int_\Omega |\nabla w_\varepsilon|^2\,dx + p|A|^{p-1}\int_\Omega |w_\varepsilon|^2\,dx\right)t_\varepsilon \\
&+ \int_\Omega |f(t_\varepsilon w_\varepsilon + A)w_\varepsilon|\,dx.
\end{aligned} \tag{7.38}$$

We now see the necessity to estimate the term $\int_\Omega |f(t_\varepsilon w_\varepsilon + A)w_\varepsilon|\,dx$. Recall that $f(z) = \lambda z + \mu z|z|^{q-1}$, we thus have

$$\begin{aligned}
|f(t_\varepsilon w_\varepsilon + A)| &\leq |\lambda|t_\varepsilon|w_\varepsilon| + |\lambda A| + 2^{q-1}\mu t_\varepsilon^q|w_\varepsilon|^q + 2^{q-1}\mu|A|^q \\
&= |\lambda|t_\varepsilon|w_\varepsilon| + 2^{q-1}\mu t_\varepsilon^q|w_\varepsilon|^q + \left(|\lambda A| + 2^{q-1}\mu|A|^q\right),
\end{aligned}$$

and hence we get

$$\begin{aligned}
&\int_\Omega |f(t_\varepsilon w_\varepsilon + A)w_\varepsilon|\,dx \\
&\qquad\leq |\lambda|\left(\int_\Omega |w_\varepsilon|^2\,dx\right)t_\varepsilon + 2^{q-1}\mu\left(\int_\Omega |w_\varepsilon|^{q+1}\,dx\right)t_\varepsilon^q \\
&\qquad\quad + \left(|\lambda A| + 2^{q-1}\mu|A|^q\right)\int_\Omega |w_\varepsilon|\,dx.
\end{aligned} \tag{7.39}$$

Put (7.39) into (7.38) we get

$$\left(\int_\Omega |w_\varepsilon|^{p+1}\,dx\right) t_\varepsilon^p \le \left(\int_\Omega |\nabla w_\varepsilon|^2\,dx + (p|A|^{p-1}+|\lambda|)\int_\Omega |w_\varepsilon|^2\,dx\right) t_\varepsilon + 2^{q-1}\mu\left(\int_\Omega |w_\varepsilon|^{q+1}\,dx\right) t_\varepsilon^q + \left(|\lambda A| + 2^{q-1}\mu|A|^q\right)\int_\Omega |w_\varepsilon|\,dx.$$

Now by the computations in appendix C, we have the following asymptotic behavior for each term appearing on the right hand side of the above expression.

$$\begin{cases} \displaystyle\int_\Omega |\nabla w_\varepsilon|^2\,dx = \frac{1}{2}K_1 + O(\sqrt{\varepsilon}\,); \\ \displaystyle\int_\Omega |w_\varepsilon|^{p+1}\,dx = \frac{1}{2}K_4 + O(\sqrt{\varepsilon}\,); \\ \displaystyle\int_\Omega |w_\varepsilon|^2\,dx = O(\varepsilon); \\ \displaystyle\int_\Omega |w_\varepsilon|^q\,dx = O(\varepsilon^{(p-q+1)/(p-1)}); \\ \displaystyle\int_\Omega |w_\varepsilon|\,dx = o(\varepsilon^{(n-2)/4}). \end{cases}$$

In particular, all of them are of $o(1)$ as $\varepsilon \to 0$ and hence we conclude the existence of positive constants $L$ and $\varepsilon_0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ we have $0 < t_\varepsilon < L$. We will see that this bound is very important for the control of $E(t_\varepsilon w_\varepsilon)$.

Recall that our goal is to estimate $E$ making use of $\Phi$. Note that $\Phi(t_\varepsilon w_\varepsilon) \le \Phi(T_\varepsilon w_\varepsilon)$. By definition of $E$ and $L$ above, we have

$$E(t_\varepsilon w_\varepsilon) \le \Phi(T_\varepsilon w_\varepsilon) + \frac{p}{2}|A|^{p-1}L^2\int_\Omega |w_\varepsilon|^2\,dx - \int_\Omega (F(t_\varepsilon w_\varepsilon + A) - F(A))\,dx. \tag{7.40}$$

And we should estimate the third term on the right. Recall that

$$F(z) = \frac{1}{2}\lambda z^2 + \frac{1}{q+1}\mu|z|^{q+1},$$

where $q+1 \in (2, 2^*)$, $\lambda$ is real and $\mu$ is nonnegative. We have

$$F(t_\varepsilon w_\varepsilon + A) - F(A) = \frac{1}{2}\lambda t_\varepsilon w_\varepsilon(t_\varepsilon w_\varepsilon + 2A) + \frac{1}{q+1}\mu\left(|t_\varepsilon w_\varepsilon + A|^{q+1} - |A|^{q+1}\right),$$

and hence

$$\begin{aligned}
&\int_{\Omega} \left(F(t_\varepsilon w_\varepsilon + A) - F(A)\right)\, dx \\
&\qquad = \frac{\lambda}{2} t_\varepsilon^2 \int_{\Omega} |w_\varepsilon|^2\, dx + \frac{\mu}{q+1} \int_{\Omega} \left(|t_\varepsilon w_\varepsilon + A|^{q+1} - |A|^{q+1}\right)\, dx.
\end{aligned}$$

Notice that $q+1 \in (2, 2^*) \subset (2,3)$, we utilize Proposition $A.1$ again to estimate the second term on the right.

$$\begin{aligned}
&\int_{\Omega} \left(F(t_\varepsilon w_\varepsilon + A) - F(A)\right)\, dx \\
\geq& \frac{\lambda}{2} t_\varepsilon^2 \int_{\Omega} |w_\varepsilon|^2\, dx + \frac{\mu}{q+1} \int_{\Omega} |t_\varepsilon w_\varepsilon|^{q+1}\, dx \\
&- \frac{\mu q}{2} |A|^{q-1} t_\varepsilon^2 \int_{\Omega} |w_\varepsilon|^2\, dx + |A|^q \int_{\Omega} t_\varepsilon w_\varepsilon\, dx \\
\geq& - \frac{|\lambda|}{2} t_\varepsilon^2 \int_{\Omega} |w_\varepsilon|^2\, dx - \frac{\mu q}{2} |A|^{q-1} t_\varepsilon^2 \int_{\Omega} |w_\varepsilon|^2\, dx \\
\geq& - \left(|\lambda| + \mu q |A|^{q-1}\right) \frac{L^2}{2} \int_{\Omega} |w_\varepsilon|^2\, dx.
\end{aligned}$$

Combining this with (7.40), we get

$$\begin{aligned}
E(t_\varepsilon w_\varepsilon) &\leq \Phi(T_\varepsilon w_\varepsilon) + \left(p|A|^{p-1} + |\lambda| + \mu q |A|^{q-1}\right) \frac{L^2}{2} \int_{\Omega} |w_\varepsilon|^2\, dx \\
&= \Phi(T_\varepsilon w_\varepsilon) + O(\varepsilon).
\end{aligned}$$

Together with (7.37) we have

$$\begin{aligned}
E(t_\varepsilon w_\varepsilon) &\leq \frac{1}{2n} S^{n/2} - \gamma' II(\varepsilon) + o(\sqrt{\varepsilon}\,) \\
&= \frac{1}{2n} S^{n/2} - \gamma' K_5 \sqrt{\varepsilon} + o(\sqrt{\varepsilon}\,) = \frac{1}{2n} S^{n/2} - \sqrt{\varepsilon}\left(\gamma' K_5 + o(1)\right).
\end{aligned}$$

Then we can choose $\varepsilon > 0$ so small that $\gamma' K_5 + o(1) > (\gamma' K_5)/2 > 0$ which implies

$$E(t_\varepsilon w_\varepsilon) \leq \frac{1}{2n} S^{n/2} - \sqrt{\varepsilon} \frac{\gamma' K_5}{2} < \frac{1}{2n} S^{n/2}.$$

We finally put $w_0 = w_\varepsilon$ and conclude that $\sup_{t \geq 0} J(t w_0) < (2n)^{-1} S^{n/2}$.

**THE CASE $n = 5$**

In this case, since $p+1 = 10/3 \in (3,4)$, we have the following inequality which is proven in appendix A.

$$|x+A|^{p+1}-|A|^{p+1}\geq$$
$$|x|^{p+1}-(p+1)|A||x|^{p}-\frac{(p+1)p(p-1)}{2}|A|^{p-1}x^{2}+(p+1)A|A|^{p-1}x,$$

for all real numbers $x$ and $A$. Again we compare $J$ and $\Phi$ via this inequality

$$\begin{aligned}J(w)\leq&\frac{1}{2}\int_{\Omega}|\nabla w|^{2}\,dx-\frac{1}{p+1}\int_{\Omega}|w|^{p+1}\,dx+|A|\int_{\Omega}|w|^{p}\,dx\\&+\frac{p(p-1)}{2}|A|^{p-1}\int_{\Omega}|w|^{2}\,dx-\int_{\Omega}(F(w+A)-F(A))\,dx\\=&\Phi(w)+|A|\int_{\Omega}|w|^{p}\,dx+\frac{p(p-1)}{2}|A|^{p-1}\int_{\Omega}|w|^{2}\,dx\\&-\int_{\Omega}(F(w+A)-F(A))\,dx=:E(w).\end{aligned}$$

We again see that $J$ is dominated by another energy $E$.

Similar to that we have done in the $n\geq 6$ case, we let $t_\varepsilon>0$ be such that $E(t_\varepsilon w_\varepsilon)=\sup_{t\geq 0}E(tw_\varepsilon)$. By similar argument as in that case, we may assume the existence of such $t_\varepsilon>0$. Consider differentiation, we get

$$\begin{aligned}&\left(\int_{\Omega}|\nabla w_\varepsilon|^{2}\,dx\right)t_\varepsilon-\left(\int_{\Omega}|w_\varepsilon|^{10/3}\,dx\right)t_\varepsilon^{7/3}+\left(\frac{7}{3}|A|\int_{\Omega}|w_\varepsilon|^{7/3}\,dx\right)t_\varepsilon^{4/3}\\&\qquad+\left(\frac{28}{9}|A|^{4/3}\int_{\Omega}|w_\varepsilon|^{2}\,dx\right)t_\varepsilon-\int_{\Omega}f(t_\varepsilon w_\varepsilon+A)w_\varepsilon\,dx=0,\end{aligned}\tag{7.41}$$

which implies

$$\begin{aligned}\left(\int_{\Omega}|w_\varepsilon|^{10/3}\,dx\right)t_\varepsilon^{7/3}\leq&\left(\int_{\Omega}|\nabla w_\varepsilon|^{2}\,dx+\frac{28}{9}|A|^{4/3}\int_{\Omega}|w_\varepsilon|^{2}\,dx\right)t_\varepsilon\\&\left(\frac{7}{3}|A|\int_{\Omega}|w_\varepsilon|^{7/3}\,dx\right)t_\varepsilon^{4/3}+\int_{\Omega}|f(t_\varepsilon w_\varepsilon+A)w_\varepsilon|\,dx.\end{aligned}\tag{7.42}$$

Again we put (7.39) into (7.42) and utilize the following asymptotic behaviors according to appendix C to conclude the existence of $L$ and $\varepsilon>0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ we have $0 < t_\varepsilon < L$.

$$\begin{cases} \int_\Omega |\nabla w_\varepsilon|^2 \, dx = \frac{1}{2}K_1 + O(\sqrt{\varepsilon}\,); \\ \int_\Omega |w_\varepsilon|^{p+1} \, dx = \frac{1}{2}K_4 + O(\sqrt{\varepsilon}\,); \\ \int_\Omega |w_\varepsilon|^2 \, dx = O(\varepsilon); \\ \int_\Omega |w_\varepsilon|^{7/3} \, dx = O(\varepsilon^{3/4}); \\ \int_\Omega |w_\varepsilon|^q \, dx = O(\varepsilon^{(p-q+1)/(p-1)}); \\ \int_\Omega |w_\varepsilon| \, dx = O(\varepsilon^{3/4}). \end{cases}$$

Thus we obtain the following estimate of $E$ involving $\Phi$.

$$\begin{aligned} E(t_\varepsilon w_\varepsilon) \leq \Phi(T_\varepsilon w_\varepsilon) &+ \left(|A|L^{7/3}\right) \int_\Omega |w_\varepsilon|^{7/3} \, dx \\ &+ \left(\frac{14}{9}|A|^{4/3}L^2\right) \int_\Omega |w_\varepsilon|^2 \, dx - \int_\Omega (F(t_\varepsilon w_\varepsilon + A) - F(A)) \, dx. \end{aligned}$$

We next estimate the third term on the right. Recall that

$$F(z) = \frac{1}{2}\lambda z^2 + \frac{1}{q+1}\mu |z|^{q+1},$$

where $q+1 \in (2, 2^*) = (2, 10/3)$, $\lambda$ is real and $\mu$ is nonnegative. If $q+1 \in (2, 3]$, we conclude as in the $n \geq 6$ case that $E(t_\varepsilon w_\varepsilon) \leq \Phi(T_\varepsilon w_\varepsilon) + O(\varepsilon)$. If $q+1 \in (3, 10/3)$, then $q+1 \in (3, 4)$, we thus apply Proposition $A.2$ again.

$$\begin{aligned} &\int_\Omega (F(t_\varepsilon w_\varepsilon + A) - F(A)) \, dx \\ &\quad = \frac{\lambda}{2} t_\varepsilon^2 \int_\Omega |w_\varepsilon|^2 \, dx + \frac{\mu}{q+1} \int_\Omega \left(|t_\varepsilon w_\varepsilon + A|^{q+1} - |A|^{q+1}\right) \, dx \\ &\quad \geq -\frac{|\lambda|}{2} t_\varepsilon^2 \int_\Omega |w_\varepsilon|^2 \, dx + \frac{\mu}{q+1} t_\varepsilon^{q+1} \int_\Omega |w_\varepsilon|^{q+1} \, dx \\ &\qquad\qquad - \mu |A| t_\varepsilon^q \int_\Omega |w_\varepsilon|^q \, dx - \mu \frac{q(q-1)}{2} |A|^{q-1} t_\varepsilon^2 \int_\Omega |w_\varepsilon|^2 \, dx \\ &\quad \geq -\frac{|\lambda|}{2} L^2 \int_\Omega |w_\varepsilon|^2 \, dx \\ &\qquad - \mu \left( |A| L^q \int_\Omega |w_\varepsilon|^q \, dx + \frac{q(q-1)}{2} |A|^{q-1} L^2 \int_\Omega |w_\varepsilon|^2 \, dx \right). \end{aligned}$$

Note by Proposition $C.7$ we have $\int_\Omega |w_\varepsilon|^q\,dx = O(\varepsilon^{(10-3q)/4})$. As we are now working with those $q \in (2, 7/3)$, we have $(10-3q) \geq 3$. Combining this with (7.41), we conclude that $E(t_\varepsilon w_\varepsilon) \leq \Phi(T_\varepsilon w_\varepsilon) + O(\varepsilon^{3/4})$.

In both cases, we have $E(t_\varepsilon w_\varepsilon) \leq \Phi(T_\varepsilon w_\varepsilon) + o(\sqrt{\varepsilon}\ )$ and hence we may use the argument at the end of $n \geq 6$ case to choose small enough $\varepsilon > 0$ to complete the proof of this case.

**THE CASE $n = 4$**

In this case $p+1 = 4$ is a natural number. We can expand the term explicitly. Namely, $(x+A)^4 - A^4 = x^4 + 4x^3A + 6x^2A^2 + 4xA^3$, for all real numbers $x$ and $A$. Thus

$$\begin{aligned} J(w) &= \frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \frac{1}{4}\int_\Omega w^4\,dx - A\int_\Omega w^3\,dx \\ &\quad - \frac{3}{2}A^2\int_\Omega w^2\,dx + A^3\int_\Omega w\,dx - \int_\Omega (F(w+A) - F(A))\,dx \\ &\leq \Phi(w) - A\int_\Omega w^3\,dx - \int_\Omega (F(w+A) - F(A))\,dx. \end{aligned}$$

We now call this term $E$ and try to give an upper bound to $\sup_{t\geq 0} E(tw_\varepsilon)$. We again let $t_\varepsilon > 0$ be such that $E(t_\varepsilon w_\varepsilon) = \sup_{t\geq 0} E(tw_\varepsilon)$. Before doing differentiation, we consider

$$E(tw_\varepsilon) = \Phi(tw_\varepsilon) - A\left(\int_\Omega w_\varepsilon^3\,dx\right)t^3 - \int_\Omega (F(tw_\varepsilon + A) - F(A))\,dx.$$

We focus on the term $\int_\Omega w_\varepsilon^3\,dx$ and notice $(w_\varepsilon)^3 = (u_\varepsilon - \delta_\varepsilon)^3 = u_\varepsilon^3 - 3u_\varepsilon^2\delta_\varepsilon + 3u_\varepsilon\delta_\varepsilon^2 - \delta_\varepsilon^3$, where $\delta_\varepsilon := \fint_\Omega u_\varepsilon\,dx$, to obtain

$$\begin{aligned} E(tw_\varepsilon) =& \Phi(tw_\varepsilon) - \int_\Omega (F(tw_\varepsilon + A) - F(A))\,dx \\ & - At^3\left(\int_\Omega u_\varepsilon^3\,dx - 3\delta_\varepsilon\int_\Omega u_\varepsilon^2\,dx + 2\delta_\varepsilon^3\int_\Omega 1\,dx\right). \end{aligned} \tag{7.43}$$

This suggest us to consider three subcases differently, namely, $A = 0$, $A > 0$ and $A < 0$. For $A = 0$, $J$ coincides with $\Phi$ and we are done in view of (7.37).

For $A > 0$, we turn (7.43) into

$$E(tw_\varepsilon) \leq \Phi(tw_\varepsilon) - \int_\Omega (F(tw_\varepsilon + A) - F(A))\ dx + \left(3A\delta_\varepsilon \int_\Omega u_\varepsilon^2\ dx\right) t^3, \quad (7.44)$$

and denote the right hand side as $E_1(tw_\varepsilon)$. We similarly let $t_\varepsilon > 0$ be such that $E_1(t_\varepsilon w_\varepsilon) = sup_{t\geq 0} E_1(tw_\varepsilon)$ and conclude the validity of assuming such $t_\varepsilon > 0$ exists. Consider differentiation

$$\begin{aligned} &\left(\int_\Omega |\nabla w_\varepsilon|^2\ dx\right) t_\varepsilon - \left(\int_\Omega |w_\varepsilon|^4\ dx\right) t_\varepsilon^3 \\ &\qquad + \left(9A\delta_\varepsilon \int_\Omega u_\varepsilon^2\ dx\right) t_\varepsilon^2 - \int_\Omega f(t_\varepsilon w_\varepsilon + A) w_\varepsilon\ dx = 0, \end{aligned} \quad (7.45)$$

which implies

$$\begin{aligned} &\left(\int_\Omega |w_\varepsilon|^4\ dx\right) t_\varepsilon^3 \\ &\leq \left(\int_\Omega |\nabla w_\varepsilon|^2\ dx\right) t_\varepsilon + \left(9A\delta_\varepsilon \int_\Omega u_\varepsilon^2\ dx\right) t_\varepsilon^2 + \int_\Omega |f(t_\varepsilon w_\varepsilon + A) w_\varepsilon|\ dx. \end{aligned} \quad (7.46)$$

We then put (7.39) into (7.46) and utilize the following asymptotic estimate for each term on the right hand side to conclude the existence of an upper bound $L$ of $t_\varepsilon$ for small enough $\varepsilon > 0$.

$$\begin{cases} \int_\Omega |\nabla w_\varepsilon|^2\ dx = \frac{1}{2} K_1 + O(\sqrt{\varepsilon}\ ); \\ \int_\Omega |w_\varepsilon|^4\ dx = \frac{1}{2} K_4 + O(\sqrt{\varepsilon}\ ); \\ \int_\Omega |w_\varepsilon|^2\ dx = O(\varepsilon\ |\log \varepsilon|); \\ \int_\Omega |w_\varepsilon|^q\ dx = O\left((\varepsilon\ |\log \varepsilon|)^{(4-q)/2}\right); \\ \int_\Omega |w_\varepsilon|\ dx = O(\varepsilon^{1/2}); \\ \int_\Omega u_\varepsilon^3\ dx = O(\sqrt{\varepsilon}\ ). \end{cases} \quad (7.47)$$

And we hence by (7.44) obtain

$$E_1(t_\varepsilon w_\varepsilon) \leq \Phi(T_\varepsilon w_\varepsilon) + \left(3A\delta_\varepsilon \int_\Omega u_\varepsilon^2\ dx\right) L^3 - \int_\Omega (F(t_\varepsilon w_\varepsilon + A) - F(A))\ dx.$$

To estimate the third term, we recall that

$$F(z) = \frac{1}{2}\lambda z^2 + \frac{1}{q+1}\mu|z|^{q+1},$$

where $q+1 \in (2, 2^*) = (2, 4)$, $\lambda$ is real and $\mu$ is nonnegative. For $q+1 \in (2, 3]$, we again argue as in the $n \geq 6$ case that

$$-\int_\Omega (F(t_\varepsilon w_\varepsilon + A) - F(A))\ dx \leq O(\varepsilon\ |\log \varepsilon|).$$

For $q+1 \in (3, 2^*] = (3, 4]$, we argue as in the $n = 5$ case that

$$\begin{aligned}\int_\Omega (F(t_\varepsilon w_\varepsilon + A) - F(A))\ dx \geq& -\frac{|\lambda|}{2}L^2 \int_\Omega |w_\varepsilon|^2\ dx \\ &- \mu\left(|A|L^q \int_\Omega |w_\varepsilon|^q\ dx + \frac{q(q+1)}{2}|A|^{q-1}L^2 \int_\Omega |w_\varepsilon|^2\ dx\right).\end{aligned}$$

Notice that $q+1 \in (3, 4)$ implies $4 - q > 1$, we have $\int_\Omega |w_\varepsilon|^q\ dx = o(\sqrt{\varepsilon}\ )$. Thus

$$-\int_\Omega F(t_\varepsilon w_\varepsilon + A) - F(A)\ dx \leq o(\sqrt{\varepsilon}\ ). \tag{7.48}$$

In both cases, we have $E_1(t_\varepsilon w_\varepsilon) \leq \Phi(T_\varepsilon w_\varepsilon) + o(\sqrt{\varepsilon}\ )$. We then choose $\varepsilon > 0$ so small that $\sup_{t\geq 0} J(tw_\varepsilon) \leq \sup_{t\geq 0} E(tw_\varepsilon) \leq \sup_{t\geq 0} E_1(tw_\varepsilon) = E_1(t_\varepsilon w_\varepsilon) < (2n)^{-1}S^{n/2}$. The proof of the subcase $A > 0$ is then completed.

For $A < 0$, we turn (7.43) into

$$\begin{aligned}E(tw_\varepsilon) = \Phi(tw_\varepsilon) &- \int_\Omega (F(tw_\varepsilon + A) - F(A))\ dx \\ &+(-A)\left(\int_\Omega u_\varepsilon^3\ dx + 2\delta_\varepsilon^3 \int_\Omega 1\ dx\right) t^3,\end{aligned}$$

and denote the right hand side as $E_2(tw_\varepsilon)$. let $t_\varepsilon > 0$ be such that $E_2(t_\varepsilon w_\varepsilon) = sup_{t\geq 0}E_2(tw_\varepsilon)$ and as what we did before, consider differentiation

$$\begin{aligned}&\left(\int_\Omega |\nabla w_\varepsilon|^2\ dx\right) t_\varepsilon - \left(\int_\Omega |w_\varepsilon|^4\ dx\right) t_\varepsilon^3 \\ &\quad + (-3A)\left(\int_\Omega u_\varepsilon^3\ dx + 2\delta_\varepsilon^3 \int_\Omega 1\ dx\right) t_\varepsilon^2 - \int_\Omega f(t_\varepsilon w_\varepsilon + A) w_\varepsilon\ dx = 0.\end{aligned}$$

Then by the asymptotic estimate (7.47), we use similar argument to the $A > 0$ case to obtain an upper bound $L > 0$ of $t_\varepsilon$ for small enough $\varepsilon > 0$. Notice that the derivation of (7.48) does not depend on the sign of $A$. Together with $\int_\Omega u_\varepsilon^3\,dx = \frac{1}{2}K_3\sqrt{\varepsilon} + o(\sqrt{\varepsilon}\,)$, we have

$$\begin{aligned} E_2(t_\varepsilon w_\varepsilon) &\le \Phi(T_\varepsilon w_\varepsilon) + \frac{(-3A)}{2}K_3\sqrt{\varepsilon} + o(\sqrt{\varepsilon}\,) \\ &= \frac{1}{2n}S^{n/2} - \gamma' II(\varepsilon) + \frac{(-3A)}{2}K_3\sqrt{\varepsilon} + o(\sqrt{\varepsilon}\,) \\ &= \frac{1}{2n}S^{n/2} - \left(\gamma' K_5 - \frac{(-3A)}{2}K_3\right)\sqrt{\varepsilon} + o(\sqrt{\varepsilon}\,). \end{aligned} \tag{7.49}$$

Now, if we choose $A_0 > 0$ such that $\gamma' K_5 - (-3A)K_3/2 > 0$, then for any $A \in (-A_0, 0)$ we have $sup_{t\ge0}J(tw_\varepsilon) \le sup_{t\ge0}E(tw_\varepsilon) \le sup_{t\ge0}E_2(tw_\varepsilon) = E_2(t_\varepsilon w_\varepsilon) < (2n)^{-1}S^{n/2}$ □

By Proposition 7.2.1, 7.3.1 and 7.4.1, Theorem 7.0.1 follows.

# Appendix A

## Some useful inequalities

**Proposition A.1.** *Suppose $r \in (2,3]$. Then we have*

$$|x+A|^r - |x|^r - |A|^r \geq -\frac{r(r-1)}{2}|A|^{r-2}x^2 + r|A|^{r-2}Ax,$$

*for all real numbers $x$ and $A$.*

*Proof.* Define $\varphi(x) := |x+A|^r - |x|^r - |A|^r$. Then we have

$$\begin{aligned}\varphi'(x) &= r\left[|x+A|^{r-2}(x+A) - |x|^{r-2}x\right], \text{ and}\\ \varphi''(x) &= r(r-1)\left[|x+A|^{r-2} - |x|^{r-2}\right].\end{aligned}$$

Note that $\varphi(0) = 0$ and $\varphi'(0) = r|A|^{r-2}A$. As $r-2 \in (0,1]$, we have

$$-|A|^{r-2} \leq |x+A|^{r-2} - |x|^{r-2} \geq |A|^{r-2}.$$

Now, for $x \geq 0$, we have

$$\begin{aligned}\varphi'(x) =& \varphi'(0) + \int_0^x \varphi''(t)\, dt\\ \geq& \varphi'(0) - r(r-1)|A|^{r-2}x\\ =& -r(r-1)|A|^{r-2}x + r|A|^{r-2}A,\end{aligned}$$

which implies

$$\varphi(x) = \varphi(0) + \int_0^x \varphi'(t)\, dt \geq -\frac{r(r-1)}{2}|A|^{r-2}x^2 + r|A|^{r-2}Ax.$$

On the other hand, for $x < 0$, we have

$$\begin{aligned}\varphi'(x) =&\varphi'(0) + \int_0^x \varphi''(t)\ dt\\ \leq&\varphi'(0) + -r(r-1)|A|^{r-2}x\\ =&-r(r-1)|A|^{r-2}x + r|A|^{r-2}A,\end{aligned}$$

which also implies

$$\varphi(x) = \varphi(0) + \int_0^x \varphi'(t)\ dt \geq -\frac{r(r-1)}{2}|A|^{r-2}x^2 + r|A|^{r-2}Ax.$$

Consequently, the inequality is true for all real numbers $x$ and $A$. $\square$

**Proposition A.2.** *Suppose $r \in (3,4)$. Then we have*

$$|x+A|^r - |x|^r - |A|^r \geq -r|A||x|^{r-1} - \frac{r(r-1)(r-2)}{2}|A|^{r-2}x^2 + r|A|^{r-2}Ax$$

*for all real numbers $x$ and $A$.*

*Proof.* Define $\varphi(x)$ as in Proposition $A$.1. Then we have the same $\varphi'(x)$ and $\varphi''(x)$ as before. Note that

$$|x+A|^{r-2} = |x|^{r-2} + (r-2)\mathrm{sgn}(x+\theta A)|x+\theta A|^{r-3}A.$$

Now, as $r-2 \in (1,2)$ and $r-3 \in (0,1)$, we have

$$\begin{aligned}\left||x+A|^{r-2} - |x|^{r-2}\right| =&(r-2)|x+\theta A|^{r-3}|A|\\ \leq&(r-2)\left(|x|+|A|\right)^{r-3}|A|\\ \leq&(r-2)\left(|A||x|^{r-3} + |A|^{r-2}\right),\end{aligned}$$

which implies

$$\varphi''(x) \geq -r(r-1)(r-2)|A||x|^{r-3} - r(r-1)(r-2)|A|^{r-2}.$$

Integrating this inequality twice as in Proposition $A$.1, the result follows. $\square$

# Appendix B

## The eigenvalue problem on $X$ with Neumann boundary condition

Recall that

$$X := \left\{ w \in X : \int_\Omega w \, dx = 0 \right\}.$$

In this section, we will prove the following proposition concerning the behavior of eigenvalues of the eigenvalue problem.

$$\begin{cases} -\Delta\varphi = \mu\varphi & \text{in } \Omega, \\ \dfrac{\partial\varphi}{\partial n} = 0 & \text{on } \partial\Omega, \\ \displaystyle\fint_\Omega \varphi \, dx = 0. \end{cases} \tag{B.1}$$

Note that by an eigenvalue, say $\mu$, to the eigenvalue problem, we mean that $\mu$ is a real number and that there exists a nonzero function $\varphi \in X$ such that for all $\psi \in X$ we have,

$$\int_\Omega \nabla\varphi \cdot \nabla\psi \, dx = \mu \int_\Omega \varphi\psi \, dx.$$

**Proposition B.1.** *Let $\Sigma$ be the set of all eigenvalues. Then we have the followings.*

*(i) $\Sigma$ is a countably infinite subset of real number;*

(ii) *We can repeat each eigenvalue according to its multiplicity, which must be finite, so that*

$$\Sigma = \{\mu_j\}_{j=1}^{\infty},$$

*where*

$$0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots \to \infty;$$

(iii) *There exists an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ of $L^2(\Omega)$, where $\varphi_j \in X$ is an eigenfunction of the eigenvalue problem corresponding to the eigenvalue $\mu_j$.*

*Proof.* For any $\varphi \in L^2(\Omega)$, denote $\Phi(\varphi)$ as the unique weak solution $u \in X \hookrightarrow L^2(\Omega)$ of the following equation.

$$\begin{cases} -\Delta u = \varphi & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \displaystyle\fint_{\Omega} u\, dx = 0. \end{cases}$$

The existence of $u \in X$ is given by Theorem 3.1.2 and the uniqueness follows from the mean zero condition together with the Green's Identity.

We first show that $\Phi$ is a compact self-adjoint operator. To see that $\Phi$ is compact, Let $\|w_j\|_2 \leq M$ be a bounded sequence in $L^2(\Omega)$. Without lost of generality, we may assume that $w_j \not\equiv 0$ for all natural number $j$. Then we have

$$\begin{aligned} \int_{\Omega} |\nabla\Phi(w_j)|^2\, dx &= \int_{\Omega} -\left(\Delta\Phi(w_j)\right)\Phi(w_j)\, dx \\ &= \int_{\Omega} w_j\Phi(w_j)\, dx \leq \|w_j\|_2 \cdot \|\Phi(w_j)\|_2. \end{aligned}$$

By the Poincare Inequality, there exists a constant $C > 0$ such that

$$\int_{\Omega} |w|^2\, dx \leq C \int_{\Omega} |\nabla w|^2\, dx,$$

for all $w \in X$. Together with $w_j \not\equiv 0$, the above equation reads

$$\|\Phi(w_j)\|_X = \|\nabla\Phi(w_j)\|_2 \leq C\|w_j\|_2 \leq CM.$$

Then Rellich compactness theorem thus guarantee that $w_j$ has convergent subsequence in $L^2(\Omega)$ and hence the compactness of $\Phi$ is verified. The self adjointness of $\Phi$ simply follows from the Green's Identity. And the result follows if we can show that $\mu_1 > 0$, which ensure no negative and zero eigenvalue to the eigenvalue problem.

Let $\mu_j$ be an eigenvalue to Problem $(B.1)$, we have

$$\int_\Omega |\nabla \varphi_j|^2 \, dx = \mu_j \int_\Omega |\varphi_j|^2 \, dx,$$

which implies $\mu_j > 0$. In particular, we have $\mu_1 > 0$. □

# Appendix C

## Computation result for $n \geq 4$

In this section, we estimate the asymptotic behavior of some integrals, which are needed in Chapter 7.

Recall in Section 7.4 we have assumed that $\Omega \subset \{x_n > 0\}$ with $0 \in \partial\Omega$ and that $\partial\Omega$ is the graph of a $C^2$-function near the origin, namely, $h : D_\delta \to \mathbb{R}$ with

$$h(x') = \frac{1}{2}\sum_{i=1}^{n-1} \alpha_i x_i^2 + o(|x'|^2) =: g(x') + o(|x'|^2),$$

where $D_\delta = \{x' \in \mathbb{R}^{n-1} : |x'| < \delta\}$ is the $\delta$ disk in $\mathbb{R}^{n-1}$ and $\alpha_i$'s are positive constants. So that we can utilize the concentration behavior of the following families of functions at the origin

$$u_\varepsilon := \varepsilon^{(n-2)/4} \frac{1}{(\varepsilon + |x|^2)^{(n-2)/2}},$$

and

$$w_\varepsilon := u_\varepsilon - \fint u_\varepsilon \, dx.$$

Note also that in this section,

$$p := \frac{n+2}{n-2} = 2^* - 1$$

and we use $C$ to denote some generic constants depending only on $n$ and $\Omega$.

**Proposition C.1.** *As $\varepsilon \to 0^+$, we have*

$$\int_\Omega |\nabla w_\varepsilon|^2 \, dx = \frac{1}{2}K_1 - I(\varepsilon) + o(\sqrt{\varepsilon}),$$

*where*

$$K_1 := (n-2)^2 \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^n} \, dy,$$

*and*

$$I(\varepsilon) := \int_{\mathbb{R}^{n-1}} \int_0^{g(x')} |\nabla w_\varepsilon|^2 \, dx_n dx' = K_2\sqrt{\varepsilon} + o(\sqrt{\varepsilon}).$$

*for some positive constant $K_2$.*

*Proof.* Note that

$$\nabla w_\varepsilon = \nabla u_\varepsilon = -(n-2)\varepsilon^{(n-2)/4} \cdot \frac{x}{(\varepsilon + |x|^2)^{n/2}},$$

which implies

$$|\nabla w_\varepsilon|^2 = (n-2)^2 \varepsilon^{(n-2)/2} \cdot \frac{|x|^2}{(\varepsilon + |x|^2)^n}.$$

Thus we have

$$\begin{aligned}
\int_{B_R(0)} |\nabla w_\varepsilon|^2 \, dx &= (n-2)^2 \varepsilon^{(n-2)/2} \int_{B_R(0)} \frac{|x|^2}{(\varepsilon + |x|^2)^n} \, dx \\
&= (n-2)^2 \varepsilon^{(n-2)/2} \frac{\sqrt{\varepsilon}^n \sqrt{\varepsilon}^2}{\varepsilon^n} \int_{B_{R/\sqrt{\varepsilon}}(0)} \frac{|y|^2}{(1+|y|^2)^n} \, dy \\
&= (n-2)^2 \int_{B_{R/\sqrt{\varepsilon}}(0)} \frac{|y|^2}{(1+|y|^2)^n} \, dy.
\end{aligned}$$

The last integral is finite, since $n \geq 3$ guarantees $2n - (2 + n - 1) \geq 2 > 1$. Similarly, we have

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B_R(0)} |\nabla w_\varepsilon|^2 \, dx &= (n-2)^2 \varepsilon^{(n-2)/2} \int_{\mathbb{R}^n \setminus B_R(0)} \frac{|x|^2}{(\varepsilon + |x|^2)^n} \, dx \\
&\leq (n-2)^2 \varepsilon^{(n-2)/2} \int_{\mathbb{R}^n \setminus B_R(0)} \frac{1}{|x|^{2n-2}} \, dx.
\end{aligned}$$

The last integral is finite, since $n \geq 3$ guarantees $(2n-2) - (n-1) \geq 2 > 1$. Thus we obtain

$$\int_{\mathbb{R}^n \setminus B_R(0)} |\nabla w_\varepsilon|^2 \, dx = O(\varepsilon^{(n-2)/2}), \tag{C.1}$$

and hence

$$\int_{B_R(0)} |\nabla w_\varepsilon|^2 \, dx = K_1 + O(\varepsilon^{(n-2)/2}). \tag{C.2}$$

Now, Let $r$ be a positive number so small that $r < \delta$ and $(\Omega \cap B_r(0)) \subset \{(x', x_n) \in \mathbb{R}^n : x_n > h(x')\}$. Denote $B_r^+(0)$ as the upper half ball, that is, $B_r(0) \cap \{x_n > 0\}$. Then we have

$$\begin{aligned}
\int_\Omega |\nabla w_\varepsilon|^2 \, dx =& \int_{B_r^+(0)} |\nabla w_\varepsilon|^2 \, dx - \int_{D_r} \int_0^{h(x')} |\nabla w_\varepsilon|^2 \, dx \\
&+ \int_{\{(x',\, x_n) \in \mathbb{R}^n : h(x') > \sqrt{1-|x|^2}\}} |\nabla w_\varepsilon|^2 \, dx + \int_{\Omega \setminus B_r(0)} |\nabla w_\varepsilon|^2 \, dx \\
=& \frac{1}{2} \int_{B_r(0)} |\nabla w_\varepsilon|^2 \, dx - \int_{\mathbb{R}^{n-1}} \int_0^{g(x')} |\nabla w_\varepsilon|^2 \, dx \\
&- \int_{D_r} \int_{g(x')}^{h(x')} |\nabla w_\varepsilon|^2 \, dx + \int_{\mathbb{R}^{n-1} \setminus D_r} \int_0^{g(x')} |\nabla w_\varepsilon|^2 \, dx \\
&+ \int_{\{(x',\, x_n) \in \mathbb{R}^n :\, h(x') > \sqrt{1-|x|^2}\}} |\nabla w_\varepsilon|^2 \, dx + \int_{\Omega \setminus B_r(0)} |\nabla w_\varepsilon|^2 \, dx
\end{aligned} \tag{C.3}$$

We shall estimate the terms in $(C.3)$ one by one. Note that by $(C.1)$ and $(C.2)$, we have

$$\frac{1}{2} \int_{B_r(0)} |\nabla w_\varepsilon|^2 \, dx = \frac{1}{2} K_1 + O(\varepsilon^{(n-2)/2}), \tag{C.4}$$

$$\int_{\mathbb{R}^{n-1} \setminus D_r} \int_0^{g(x')} |\nabla w_\varepsilon|^2 \, dx_n dx' \le \int_{\mathbb{R}^n \setminus B_r(0)} |\nabla w_\varepsilon|^2 \, dx = O(\varepsilon^{(n-2)/2}), \tag{C.5}$$

$$\int_{\{(x',\, x_n) \in \mathbb{R}^n :\, h(x') > \sqrt{1-|x|^2}\}} |\nabla w_\varepsilon|^2 \, dx_n dx' \le \int_{\mathbb{R}^n \setminus B_r(0)} |\nabla w_\varepsilon|^2 \, dx = O(\varepsilon^{(n-2)/2}), \tag{C.6}$$

$$\int_{\Omega \setminus B_r(0)} |\nabla w_\varepsilon|^2 \, dx \le \int_{\mathbb{R}^n \setminus B_r(0)} |\nabla w_\varepsilon|^2 \, dx = O(\varepsilon^{(n-2)/2}), \tag{C.7}$$

Combining $(C.3)$ to $(C.7)$, we have

$$\begin{aligned}
\int_\Omega |\nabla w_\varepsilon|^2 \, dx =& \frac{1}{2} K_1 - \int_{\mathbb{R}^{n-1}} \int_0^{g(x')} |\nabla w_\varepsilon|^2 \, dx_n dx' \\
&- \int_{D_r} \int_{g(x')}^{h(x')} |\nabla w_\varepsilon|^2 \, dx_n dx' + O(\varepsilon^{(n-2)/2}).
\end{aligned} \tag{C.8}$$

So we have to compute,

$$
\begin{aligned}
&\int_{\mathbb{R}^{n-1}}\int_0^{g(x')} |\nabla w_\varepsilon|^2 \, dx_n dx' \\
&= (n-2)^2\varepsilon^{(n-2)/2}\int_{\mathbb{R}^{n-1}}\int_0^{g(x')} \frac{|x|^2}{(\varepsilon+|x|^2)^n}\, dx_n dx' \\
&= (n-2)^2\varepsilon^{(n-2)/2}\frac{\sqrt{\varepsilon}^2\sqrt{\varepsilon}^n}{\varepsilon^n}\int_{\mathbb{R}^{n-1}}\int_0^{g(x')} \frac{|x\,/\sqrt{\varepsilon}|^2}{\left(1+|x\,/\sqrt{\varepsilon}|^2\right)^n}\, d\left(\frac{x_n}{\sqrt{\varepsilon}}\right) d\left(\frac{x'}{\sqrt{\varepsilon}}\right) \\
&= (n-2)^2\int_{\mathbb{R}^{n-1}}\int_0^{\sqrt{\varepsilon}g(y')} \frac{|y|^2}{(1+|y|^2)^n}\, dy_n dy'.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\frac{\displaystyle\int_{\mathbb{R}^{n-1}}\int_0^{g(x')} |\nabla w_\varepsilon|^2 \, dx_n dx'}{\sqrt{\varepsilon}} \\
&\qquad = 2\sqrt{\varepsilon}(n-2)^2\int_{\mathbb{R}^{n-1}} \frac{|y'|^2+\varepsilon|g(y)'|^2}{(1+|y'|^2+\varepsilon|g(y')^2|)^n}\cdot\frac{g(y')}{2\sqrt{\varepsilon}}\, dy' \\
&\qquad = (n-2)^2\int_{\mathbb{R}^{n-1}} \frac{|y'|^2 g(y')}{(1+|y'|^2)^n}\, dy' + o(1) \\
&\qquad =: K_2 + o(1),
\end{aligned}
$$

where $K_2$ is a positive constant. The finiteness of $K_2$ follows from $2n-(2+2+n-2)\geq 2>1$ and $n\geq 4$. Consequently,

$$
I(\varepsilon) := \int_{\mathbb{R}^{n-1}}\int_0^{g(x')} |\nabla w_\varepsilon|^2 \, dx_n dx' = K_2\sqrt{\varepsilon}+o(\sqrt{\varepsilon}). \tag{C.9}
$$

Finally, we compute

$$
\left|\int_{D_r}\int_{g(x')}^{h(x')} |\nabla w_\varepsilon|^2 \, dx_n dx'\right| \leq (n-2)^2\varepsilon^{(n-2)/2}\int_{D_r}\frac{|h(x')-g(x')|}{(\varepsilon+|x'|^2)^{n-1}}\, dx'.
$$

Recall that $\Omega$ and $r$ are fixed and that $|h(x')-g(x)'|=o(|x'|^2)$ as $|x'|\to 0$. We can choose $\iota>0$ so small that

$$
|h(x')-g(x')|\leq \sigma|x'|^2,
$$

for all $x'\in D_\iota\subset D_r$. Then by compactness of $\overline{D_r}\setminus D_\iota$, we can choose a positive constant $M$, depending only on $\Omega$, $\sigma$ and $r$, such that

$$
|h(x')-g(x')|\leq M|x'|^2,
$$

for all $x' \in \overline{D_r} \setminus D_\iota$. Thus we can choose a positive constant $C(\sigma)$ so big that

$$M \le C(\sigma)|x'|^{1/2},$$

for all $x'$ on $\overline{D_r} \setminus D_\iota$. Note that $C(\sigma)$ above indeed depends on $\Omega$ and $r$, but we do not emphasize the dependence as they are already fixed in this context. Consequently, we have

$$|h(x') - g(x')| \le \sigma|x'|^2 + C(\sigma)|x'|^{5/2},$$

for all $x'$ on $\overline{D_r}$. Therefore,

$$\begin{aligned}
&\left|\int_{D_r}\int_{g(x')}^{h(x')} |\nabla w_\varepsilon|^2 \, dx_n dx'\right| \\
&\le (n-2)^2 \varepsilon^{(n-2)/2} \int_{D_r} \frac{\sigma|x'|^2 + C(\sigma)|x'|^{5/2}}{(\varepsilon + |x'|^2)^{n-1}} \, dx' \\
&\le (n-2)^2 \varepsilon^{(n-2)/2} \cdot \frac{\sqrt{\varepsilon}^{n-1}\sqrt{\varepsilon}^2}{\varepsilon^{n-1}} \int_{D_r} \frac{\sigma \left|x' / \sqrt{\varepsilon}\right|^2 + C(\sigma)\left|x' / \sqrt{\varepsilon}\right|^{5/2} \varepsilon^{1/4}}{\left(1 + |x' / \sqrt{\varepsilon}|^2\right)^{n-1}} \, dx' \\
&\le (n-2)^2 \sqrt{\varepsilon} \int_{D_{r/\sqrt{\varepsilon}}} \sigma \frac{|y'|^2}{(1+|y'|^2)^{n-1}} + C(\sigma)\varepsilon^{1/4} \frac{|y'|^{5/2}}{(1+|y'|^2)^{n-1}} \, dy' \\
&\le C\sqrt{\varepsilon}\left(\sigma + C(\sigma)\varepsilon^{1/4}\right),
\end{aligned}$$

where $C$ is a positive constant depending only on $n$. Note that $C$ is finite, since $2n-2-(2+n-2) \ge 2 > 1$, $2n-2-(5/2+n-2) \ge 3/2 > 1$ and $n \ge 4$. Then

$$\limsup_{\varepsilon \to 0^+} \frac{\left|\int_{D_r}\int_{g(x')}^{h(x')} |\nabla w_\varepsilon|^2 \, dx_n dx'\right|}{\sqrt{\varepsilon}} \le C\sigma.$$

As $\sigma$ is arbitrary, we conclude that

$$\left|\int_{D_r}\int_{g(x')}^{h(x')} |\nabla w_\varepsilon|^2 \, dx_n dx'\right| = o(\sqrt{\varepsilon}). \tag{C.10}$$

The desired result follows from $(C.8)$, $(C.9)$ and $(C.10)$. □

**Proposition C.2.** *As $\varepsilon \to 0^+$, we have*

$$\int_\Omega |u_\varepsilon|^p \, dx = \frac{1}{2}K_3\varepsilon^{(n-2)/4} + o(\varepsilon^{(n-2)/4}),$$

*where*

$$K_3 := \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{(n+2)/2}}\, dy.$$

*Proof.* Notice by a similar argument to $(C.3)$, we have

$$\begin{aligned}
\int_\Omega |u_\varepsilon|^p\, dx = \frac{1}{2}\int_{B_r(0)} |u_\varepsilon|^p\, dx - \int_{\mathbb{R}^{n-1}}\int_0^{g(x')} |u_\varepsilon|^p\, dx \\
- \int_{D_r}\int_{g(x')}^{h(x')} |u_\varepsilon|^p\, dx + \int_{\mathbb{R}^{n-1}\setminus D_r}\int_0^{g(x')} |u_\varepsilon|^p\, dx \\
+ \int_{\{(x',\, x_n)\in\mathbb{R}^n:\ h(x')>\sqrt{1-|x|^2}\}} |u_\varepsilon|^p\, dx + \int_{\Omega\setminus B_r(0)} |u_\varepsilon|^p\, dx,
\end{aligned} \tag{C.11}$$

where $r$ is chosen as in Proposition $C.1$. We first estimate

$$\begin{aligned}
\int_{B_r(0)} |u_\varepsilon|^p\, dx &= \left[\varepsilon^{(n-2)/4}\right]^{(n+2)/(n-2)} \int_{B_r(0)} \frac{1}{\left[(\varepsilon+|x|^2)^{(n-2)/2}\right]^{(n+2)/(n-2)}}\, dx \\
&= \varepsilon^{(n+2)/4} \int_{B_r(0)} \frac{1}{(\varepsilon+|x|^2)^{(n+2)/2}}\, dx \\
&= \varepsilon^{(n+2)/4} \frac{\sqrt{\varepsilon}^n}{\varepsilon^{(n+2)/2}} \int_{B_r(0)} \frac{1}{\left(1+|x\,/\sqrt{\varepsilon}|^2\right)^{(n+2)/2}}\, d(\frac{x}{\sqrt{\varepsilon}}) \\
&= \varepsilon^{(n-2)/4} \int_{B_{r/\sqrt{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^{(n+2)/2}}\, dy.
\end{aligned}$$

Note that

$$\frac{\int_{B_r(0)} |u_\varepsilon|^p\, dx}{\varepsilon^{(n-2)/4}} = \int_{B_{r/\sqrt{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^{(n+2)/2}}\, dy \longrightarrow \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{(n+2)/2}}\, dy =: K_3,$$

and $K_3$ is a finite number since $(n+2)-(n-1) = 3 > 1$. Therefore we have

$$\frac{1}{2}\int_{B_r(0)} |u_\varepsilon|^p dx = \frac{1}{2}K_3\varepsilon^{(n-2)/4} + o(\varepsilon^{(n-2)/4}). \tag{C.12}$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus B_r(0)} |u_\varepsilon|^p\, dx &= \varepsilon^{(n+2)/4} \int_{\mathbb{R}^n\setminus B_r(0)} \frac{1}{(\varepsilon+|x|^2)^{(n+2)/2}}\, dx \\
&\le \varepsilon^{(n+2)/4} \int_{\mathbb{R}^n\setminus B_r(0)} \frac{1}{|x|^{n+2}}\, dx,
\end{aligned}$$

which implies

$$\int_{\mathbb{R}^n \setminus B_r(0)} |u_\varepsilon|^p dx = O(\varepsilon^{\frac{n+2}{4}}), \tag{C.13}$$

since $(n+2)-(n-1)=3>1$. Note that $(C.13)$ implies the last three terms of $(C.11)$ is of $O(\varepsilon^{(n+2)/4})$. Next, we estimate

$$\begin{aligned}
&\int_{D_r}\int_0^{g(x')} |u_\varepsilon|^p \, dx_n dx' \\
&\le \varepsilon^{(n+2)/4} \int_{\mathbb{R}^{n-1}}\int_0^{g(x')} \frac{1}{(\varepsilon+|x|^2)^{(n+2)/2}} \, dx_n dx' \\
&= \varepsilon^{(n+2)/4} \frac{\sqrt{\varepsilon}^n}{\varepsilon^{(n+2)/2}} \int_{\mathbb{R}^{n-1}}\int_0^{g(x')} \frac{1}{(1+|x\ /\sqrt{\varepsilon}|^2)^{(n+2)/2}} \, d(\frac{x}{\sqrt{\varepsilon}}) d(\frac{x'}{\sqrt{\varepsilon}}) \\
&= \varepsilon^{(n+2)/4} \int_{\mathbb{R}^{n-1}}\int_0^{\sqrt{\varepsilon} g(y')} \frac{1}{(1+|y|^2)^{(n+2)/2}} \, dy_n dy' \\
&\le \varepsilon^{(n+2)/4} \int_{\mathbb{R}^{n-1}} \frac{\sqrt{\varepsilon} g(y')}{(1+|y'|^2)^{(n+2)/2}} \, dy' \\
&\le C\varepsilon^{n/4} \int_{\mathbb{R}^{n-1}} \frac{|y'|^2}{(1+|y'|^2)^{(n+2)/4}} \, dy'.
\end{aligned}$$

Thus we obtain

$$\int_{D_r}\int_0^{g(x')} |u_\varepsilon|^p \, dx_n dx' = O(\varepsilon^{n/4}), \tag{C.14}$$

since $(n+2)-(2+n-2)=2>1$. Finally, we compute

$$\begin{aligned}
\left| \int_{D_r}\int_{g(x')}^{h(x')} |u_\varepsilon|^p \, dx_n dx' \right| &\le \varepsilon^{(n+2)/4} \int_{D_r} \left| \int_{g(x')}^{h(x')} \frac{1}{(\varepsilon+|x|^2)^{(n+2)/2}} \, dx_n \right| \, dx' \\
&\le \varepsilon^{(n+2)/4} \int_{\mathbb{R}^{n-1}}\int_{g(x')}^{h(x')} \frac{1}{(\varepsilon+|x'|^2)^{(n+2)/2}} |dx_n| \, dx' \\
&\le \varepsilon^{(n+2)/4} \int_{\mathbb{R}^{n-1}} \frac{|h(x')-g(x')|}{(\varepsilon+|x'|^2)^{(n+2)/2}} \, dx' \\
&\le C\varepsilon^{(n+2)/4} \int_{\mathbb{R}^{n-1}} \frac{|x'|^2}{(\varepsilon+|x'|^2)^{(n+2)/2}} \, dx' \\
&= C\varepsilon^{(n+2)/4} \frac{\sqrt{\varepsilon}^2 \sqrt{\varepsilon}^{n-1}}{\varepsilon^{(n+2)/2}} \int_{\mathbb{R}^{n-1}} \frac{|x'\ /\sqrt{\varepsilon}|^2}{(1+|x'\ /\sqrt{\varepsilon}|^2)^{(n+2)/2}} \, d(\frac{x'}{\sqrt{\varepsilon}}) \\
&\le C\varepsilon^{n/4} \int_{\mathbb{R}^{n-1}} \frac{|y'|^2}{(1+|y'|^2)^{(n+2)/2}} \, dy',
\end{aligned}$$

and get

$$\left| \int_{D_r}\int_{g(x')}^{h(x')} |u_\varepsilon|^p \, dx_n dx' \right| = O(\varepsilon^{n/4}). \tag{C.15}$$

Combining $(C.12)$ to $(C.15)$ into $(C.11)$, we obtain the desired result. □

**Proposition C.3.** *As $\varepsilon \to 0^+$, we have*

$$\int_\Omega u_\varepsilon \, dx = O(\varepsilon^{(n-2)/4}),$$

*and hence*

$$\delta_\varepsilon := \fint_\Omega u_\varepsilon \, dx = O(\varepsilon^{(n-2)/4}).$$

*Proof.* Notice that the idea of $(C.3)$ does not work here, we consider the following

$$\begin{aligned}\int_\Omega u_\varepsilon \, dx =& \frac{1}{2}\int_{B_r(0)} u_\varepsilon \, dx - \int_{D_r}\int_0^{h(x')} u_\varepsilon \, dx \\ &+ \int_{\{(x',\, x_n)\in\mathbb{R}^n:\ h(x')>\sqrt{1-|x|^2}\}} u_\varepsilon \, dx + \int_{\Omega\setminus B_r(0)} u_\varepsilon \, dx.\end{aligned} \tag{C.16}$$

We then estimate

$$\begin{aligned}\int_{B_r(0)} u_\varepsilon \, dx =& \varepsilon^{(n-2)/4}\int_{B_r(0)} \frac{1}{(\varepsilon+|x|^2)^{(n-2)/2}} \, dx \\ =& \varepsilon^{(n-2)/4}\int_0^r \frac{\omega_n \rho^{n-1}}{(\varepsilon+\rho^2)^{(n-2)/2}} \, d\rho.\end{aligned}$$

Note that

$$\lim_{\varepsilon\to 0}\frac{\displaystyle\int_{B_r(0)} u_\varepsilon \, dx}{\varepsilon^{(n-2)/4}} = \lim_{\varepsilon\to 0}\int_0^r \frac{\omega_n\rho^{n-1}}{(\varepsilon+\rho^2)^{(n-2)/2}} \, d\rho = \int_0^r \omega_n \rho \, d\rho = \frac{\omega_n r^2}{2}.$$

Hence we have

$$\frac{1}{2}\int_{B_r(0)} u_\varepsilon \, dx = O(\varepsilon^{(n-2)/4}) \tag{C.17}$$

and

$$\int_{B_R(0)\setminus B_r(0)} u_\varepsilon \, dx = O(\varepsilon^{(n-2)/4}), \tag{C.18}$$

as $r, R$ are fixed. Note that $(C.18)$ implies the last two terms of $(C.16)$ is of

$O(\varepsilon^{(n-2)/4})$. We next compute

$$\begin{aligned}\int_{D_r}\int_0^{h(x')} u_\varepsilon \, dx_n dx' =& \varepsilon^{(n-2)/4}\int_{D_r}\int_0^{h(x')} \frac{1}{(\varepsilon+|x|^2)^{(n-2)/2}} \, dx_n dx' \\ \leq& \varepsilon^{(n-2)/4}\int_{D_r} \frac{h(x')}{(\varepsilon+|x'|^2)^{(n-2)/2}} \, dx' \\ \leq& C\varepsilon^{(n-2)/4}\int_{D_r} \frac{|x'|^2}{(\varepsilon+|x'|^2)^{(n-2)/2}} \, dx' \\ \leq& C\varepsilon^{(n-2)/4}\int_0^r \frac{\rho^2\rho^{n-2}}{(\varepsilon+|\rho|^2)^{(n-2)/2}} \, d\rho \\ \leq& C\varepsilon^{(n-2)/4}\int_0^r \rho^2 \, d\rho,\end{aligned}$$

and get

$$\int_{D_r}\int_0^{h(x')} u_\varepsilon \, dx_n dx' = O(\varepsilon^{(n-2)/4}). \tag{C.19}$$

Then the desired result follows from $(C.16)$ to $(C.19)$. □

**Proposition C.4.** *As $\varepsilon \to 0^+$, we have*

$$\int_\Omega |w_\varepsilon|^{p+1} \, dx = \frac{1}{2}K_4 - II(\varepsilon) + o(\sqrt{\varepsilon}),$$

*where*

$$K_4 := \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^n} \, dy,$$

*and*

$$II(\varepsilon) := \int_{\mathbb{R}^{n-1}}\int_0^{g(x')} |w_\varepsilon|^{p+1} \, dx_n dx' = K_5\sqrt{\varepsilon} + o(\sqrt{\varepsilon}).$$

*for some positive constant $K_5$.*

**Remark C.5.** From [2], we have

$$\frac{K_1}{K_4^{2/(p+1)}} = \frac{K_1}{K_4^{(n-2)/n}} = S,$$

where $S$ is the best constant of the Sobolev inequality

$$S\left[\int_{\mathbb{R}^n} |w|^{p+1} \, dx\right]^{2/(p+1)} \leq \int_{\mathbb{R}^n} |\nabla w|^2 \, dx.$$

*Proof.* Note that $|w_\varepsilon|^{p+1} = |u_\varepsilon - \delta_\varepsilon|^{p+1} = |u_\varepsilon|^{p+1} - (p+1)(u_\varepsilon - \theta\delta_\varepsilon)|u_\varepsilon - \theta\delta_\varepsilon|^{p-1}\delta_\varepsilon$, for some $\theta \in [0,1]$.

$$\begin{aligned}
&\left|\int_\Omega |w_\varepsilon|^{p+1}\,dx - \int_\Omega |u_\varepsilon|^{p+1}\,dx\right| \\
&\qquad \le (p+1)\int_\Omega |u_\varepsilon - \theta\delta_\varepsilon|^p \delta_\varepsilon\,dx \\
&\qquad \le C\delta_\varepsilon \int_\Omega (|u_\varepsilon| + \delta_\varepsilon)^p\,dx \\
&\qquad \le C\delta_\varepsilon \int_\Omega |u_\varepsilon|^p\,dx + C\delta_\varepsilon^{p+1} \\
&\qquad = O(\varepsilon^{(n-2)/4})\cdot O(\varepsilon^{(n-2)/4}) + \left[O(\varepsilon^{(n-2)/4})\right]^{p+1} \\
&\qquad = O(\varepsilon^{(n-2)/2}) = O(\varepsilon),
\end{aligned}$$

since $n \ge 4$ implies $(n-2)/2 \ge 1$. We then have

$$\int_\Omega |w_\varepsilon|^{p+1}\,dx = \int_\Omega |u_\varepsilon|^{p+1} + O(\varepsilon). \tag{C.20}$$

This suggest us to compute $\int_\Omega |u_\varepsilon|^{p+1}dx$ using the idea of $(C.3)$. As before, we first estimate

$$\begin{aligned}
\int_{B_r(0)} |u_\varepsilon|^{p+1}\,dx &= \left[\varepsilon^{(n-2)/4}\right]^{2n/(n-2)} \int_{B_r(0)} \frac{1}{[(\varepsilon + |x|^2)^{(n-2)/2}]^{2n/(n-2)}}\,dx \\
&= \varepsilon^{n/2}\int_{B_r(0)} \frac{1}{(\varepsilon + |x|^2)^n}\,dx \\
&= \varepsilon^{n/2}\frac{\sqrt{\varepsilon}^n}{\varepsilon^n}\int_{B_r(0)} \frac{1}{(1 + |x\,/\sqrt{\varepsilon}|^2)^n}\,d(\frac{x}{\sqrt{\varepsilon}}) \\
&= \int_{B_{r/\sqrt{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^n}\,dy \longrightarrow \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^n}\,dy =: K_4.
\end{aligned} \tag{C.21}$$

Note that $K_4$ is finite since $n \ge 3$ implies $2n - (n-1) \ge 4 > 1$. To compute the order of convergence above, we consider

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus B_r(0)} |u_\varepsilon|^{p+1}\,dx &= \varepsilon^{n/2}\int_{\mathbb{R}^n\setminus B_r(0)} \frac{1}{(\varepsilon + |x|^2)^n}\,dx \\
&= \int_{\mathbb{R}^n\setminus B_{r/\sqrt{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^n}\,dy \\
&= \int_{r/\sqrt{\varepsilon}}^\infty \frac{\omega_n \rho^{n-1}}{(1+\rho^2)^n}\,d\rho =: \psi(\varepsilon).
\end{aligned} \tag{C.22}$$

Observe that

$$\psi'(\varepsilon) = \frac{-\omega_n(r/\sqrt{\varepsilon})^{n-1}}{(1+(r/\sqrt{\varepsilon})^2)^n}\cdot(-\frac{1}{2}\cdot\frac{r}{\varepsilon^{3/2}}) = \varepsilon^{(n-2)/2}\frac{\omega_n r^n}{2(\varepsilon+r^2)^n}, \tag{C.23}$$

which implies

$$\lim_{\varepsilon\to 0}\frac{\psi(\varepsilon)}{\varepsilon^{n/2}} = \lim_{\varepsilon\to 0}\frac{2\psi'(\varepsilon)}{n\varepsilon^{(n-2)/2}} = \lim_{\varepsilon\to 0}\frac{\omega_n r^n}{n(\varepsilon+r^2)^n} = \frac{\omega_n}{nr^n}. \tag{C.24}$$

Thus by $(C.22)$ to $(C.24)$, we have

$$\int_{\mathbb{R}^n\setminus B_r(0)} |u_\varepsilon|^{p+1}\,dx = O(\varepsilon^{n/2}), \tag{C.25}$$

and hence we obtain the desired order of convergence

$$\begin{aligned}\int_{B_r(0)} |u_\varepsilon|^{p+1}\,dx &= \int_{\mathbb{R}^n} |u_\varepsilon|^{p+1}\,dx - \int_{\mathbb{R}^n\setminus B_r(0)} |u_\varepsilon|^{p+1}\,dx \\ &= \int_{\mathbb{R}^n}\frac{1}{(1+|y|^2)^n}\,dy + O(\varepsilon^{n/2}) =: K_4 + O(\varepsilon^{n/2}).\end{aligned} \tag{C.26}$$

Note that $(C.25)$ implies the last three terms of the corresponding version of $(C.3)$ is of $O(\varepsilon^{n/2})$.

We next compute

$$\begin{aligned}II(\varepsilon) :=& \int_{\mathbb{R}^{n-1}}\int_0^{g(x')} |u_\varepsilon|^{p+1}\,dx_n dx' \\ =&\varepsilon^{n/2}\frac{\sqrt{\varepsilon}^n}{\varepsilon^n}\int_{\mathbb{R}^{n-1}}\int_0^{g(x')}\frac{1}{(1+|x\,/\sqrt{\varepsilon}|^2)^n}\,d(\frac{x_n}{\sqrt{\varepsilon}})d(\frac{x'}{\sqrt{\varepsilon}}) \\ =&\int_{\mathbb{R}^{n-1}}\int_0^{\sqrt{\varepsilon}g(x')}\frac{1}{(1+|y|^2)^n}\,dy_n dy'.\end{aligned}$$

Note that

$$II'(\varepsilon) = \int_{\mathbb{R}^{n-1}}\frac{1}{(1+|y'|^2+\varepsilon|g(y')|^2)^n}\cdot(\frac{1}{2\sqrt{\varepsilon}}g(y'))\,dy',$$

which implies

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{II(\varepsilon)}{\sqrt{\varepsilon}} = \lim_{\varepsilon\to 0}2\sqrt{\varepsilon}II'(\varepsilon) &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^{n-1}}\frac{g(y')}{(1+|y'|^2+\varepsilon|g(y')|^2)^n}\,dy' \\ &= \int_{\mathbb{R}^{n-1}}\frac{g(y')}{(1+|y'|^2)^n}\,dy' =: K_5.\end{aligned}$$

The constant $K_5$ is finite, since $n \geq 3$ implies $2n - (2 + n - 2) \geq 3 > 1$. Consequently, we obtain

$$II(\varepsilon) = K_5\sqrt{\varepsilon} + o(\sqrt{\varepsilon}). \tag{C.27}$$

Finally, for each $\sigma > 0$, let $C_\sigma$ be as in Proposition $C.1$. We then compute

$$\begin{aligned}
\left|\int_{D_r}\int_{g(x')}^{h(x')} |u_\varepsilon|^{p+1}\, dx_n dx'\right| &= \varepsilon^{n/2}\left|\int_{D_r}\int_{g(x')}^{h(x')} \frac{1}{(\varepsilon + |x|^2)^n}\, dx_n dx'\right| \\
&\leq \varepsilon^{n/2}\int_{\mathbb{R}^{n-1}} \frac{|h(x') - g(x')|}{(\varepsilon + |x'|^2)^n}\, dx' \\
&\leq \varepsilon^{n/2}\int_{\mathbb{R}^{n-1}} \frac{\sigma|x'|^2 + C(\sigma)|x'|^{5/2}}{(\varepsilon + |x'|^2)^n}\, dx' \\
&= \varepsilon^{n/2}\frac{\sqrt{\varepsilon}^2\sqrt{\varepsilon}^{n-1}}{\varepsilon^n}\int_{\mathbb{R}^{n-1}} \frac{\sigma\left|x' / \sqrt{\varepsilon}\right|^2 + C(\sigma)\varepsilon^{1/4}\left|x' / \sqrt{\varepsilon}\right|^{5/2}}{1 + \left|x' / \sqrt{\varepsilon}\right|^2}\, d(\frac{x'}{\sqrt{\varepsilon}}) \\
&= \sqrt{\varepsilon}\int_{\mathbb{R}^{n-1}} \frac{\sigma|y'|^2 + C(\sigma)\varepsilon^{1/4}|y'|^{5/2}}{(1 + |y'|^2)^n}\, dy' \\
&\leq C\sqrt{\varepsilon}\left(\sigma + C(\sigma)\varepsilon^{1/4}\right).
\end{aligned}$$

Since $\sigma > 0$ is arbitrary, we get

$$\int_{D_r}\int_{g(x')}^{h(x')} |u_\varepsilon|^{p+1}\, dx_n dx' = o(\sqrt{\varepsilon}). \tag{C.28}$$

Now, by the corresponding version of $(C.3)$, $(C.20)$ and $(C.25)$ to $(C.28)$, we obtain

$$\int_\Omega |w_\varepsilon|^{p+1}\, dx = \frac{1}{2}K_4 - II(\varepsilon) + o(\sqrt{\varepsilon})$$

and complete the proof. □

**Proposition C.6.** *As $\varepsilon \to 0^+$, we have*

$$\int_\Omega w_\varepsilon^2\, dx = \begin{cases} K_3\varepsilon + o(\varepsilon) & \text{for } n \geq 5, \\ K_3\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|) & \text{for } n = 4, \end{cases}$$

*where*

$$K_3 = \begin{cases} \displaystyle\int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}}\, dy & \text{for } n \geq 5, \\ \dfrac{\omega_4}{2} & \text{for } n = 4. \end{cases}$$

*Proof.* Note that by Proposition $C.3$, we have

$$\int_\Omega w_\varepsilon^2\,dx = \int_\Omega (u_\varepsilon - \delta_\varepsilon)^2\,dx = \int_\Omega u_\varepsilon^2\,dx - \delta_\varepsilon^2|\Omega| = \int_\Omega u_\varepsilon^2\,dx + O(\varepsilon^{(n-2)/2}). \quad \text{(C.29)}$$

This suggests us to estimate $\int_\Omega u_\varepsilon^2\,dx$ by $(C.16)$. For $n \geq 5$, consider

$$\begin{aligned}\int_{B_r(0)} u_\varepsilon^2\,dx =& \varepsilon^{(n-2)/2}\frac{\sqrt{\varepsilon}^n}{\varepsilon^{n-2}}\int_{B_{r/\sqrt{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^{n-2}}\,dy\\ =& \varepsilon\int_{B_{r/\sqrt{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^{n-2}}\,dy,\end{aligned}$$

which implies

$$\int_{B_r(0)} u_\varepsilon^2\,dx = K_3\varepsilon + o(\varepsilon). \quad \text{(C.30)}$$

Note that $K_3$ is finite since $n \geq 5$ implies $2(n-2)-(n-1) \geq 2 > 1$. For $n = 4$, consider

$$\begin{aligned}\int_{B_r(0)} u_\varepsilon^2\,dx =& \varepsilon\int_0^r \frac{\omega_4\rho^3}{(\varepsilon+\rho^2)^2}\,d\rho = \frac{\omega_4\varepsilon}{2}\int_0^r \frac{\rho^2}{(\varepsilon+\rho^2)^2}\,d\rho^2\\ =& \frac{\omega_4\varepsilon}{2}\int_0^r \frac{1}{\varepsilon+\rho^2}\,d\rho^2 + \frac{\omega_4\varepsilon^2}{2}\int_0^r \frac{-1}{(\varepsilon+\rho^2)^2}\,d\rho^2\\ =& \frac{\omega_4\varepsilon}{2}\left[\log(\varepsilon+r^2) - \log\varepsilon\right] + \frac{\omega_4\varepsilon^2}{2}\left[\frac{1}{\varepsilon+r^2} - \frac{1}{\varepsilon}\right],\end{aligned}$$

which implies

$$\int_{B_r(0)} u_\varepsilon^2\,dx = \frac{\omega_4}{2}\varepsilon|\log\varepsilon| + o(\varepsilon|\log\varepsilon|). \quad \text{(C.31)}$$

We next compute

$$\begin{aligned}\int_{B_R(0)\setminus B_r(0)} u_\varepsilon^2\,dx =& \varepsilon^{(n-2)/2}\int_{B_R(0)\setminus B_r(0)} \frac{1}{(\varepsilon+|x|^2)^{n-2}}\,dx\\ \leq& \varepsilon^{(n-2)/2}\int_{B_R(0)\setminus B_r(0)} \frac{1}{|x|^{2n-4}}\,dx,\end{aligned}$$

and hence

$$\int_{B_R(0)\setminus B_r(0)} u_\varepsilon^2\,dx = \begin{cases} O(\varepsilon^{3/2}) & \text{for } n \geq 5,\\ O(\varepsilon) & \text{for } n = 4.\end{cases} \quad \text{(C.32)}$$

Finally, we compute

$$\begin{aligned}\int_{D_r}\int_0^{h(x')} u_\varepsilon^2\, dx_n dx' =&\varepsilon^{(n-2)/2}\int_{D_r}\int_0^{h(x')} \frac{1}{(\varepsilon+|x|^2)^{n-2}}\, dx_n dx' \\ \leq&\varepsilon^{(n-2)/2}\int_{D_r}\int_0^{h(x')} \frac{1}{(\varepsilon+|x'|^2)^{n-2}}\, dx_n dx' \\ \leq&C\varepsilon^{(n-2)/2}\int_{D_r} \frac{|x'|^2}{(\varepsilon+|x'|^2)^{n-2}}\, dx' \\ =&C\varepsilon^{3/2}\int_0^{r/\sqrt{\varepsilon}} \frac{|y'|^2}{(1+|y'|^2)^{n-2}}\, dy' =: C\varepsilon^{3/2}\varphi(\varepsilon).\end{aligned}$$

Note that

$$\varphi'(\varepsilon) = \frac{(r/\sqrt{\varepsilon})^2(r/\sqrt{\varepsilon})^{n-1}}{(1+(r/\sqrt{\varepsilon})^2)^{n-2}}\cdot\frac{r}{-2\varepsilon^{3/2}} = \frac{r^{n+1}}{(\varepsilon+r^2)^{n-2}}\cdot\varepsilon^{(n-4)/2}\cdot\frac{\varepsilon^{-3/2}}{-2},$$

which implies

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon^{1/2}\varphi(\varepsilon) =&\lim_{\varepsilon\to 0}\frac{\varphi(\varepsilon)}{\varepsilon^{-1/2}} = \lim_{\varepsilon\to 0}\frac{-2\varphi'(\varepsilon)}{\varepsilon^{-3/2}} \\ =&\lim_{\varepsilon\to 0}\left[\frac{r^{n+1}}{(\varepsilon+r^2)^{n-2}}\cdot\varepsilon^{(n-4)/2}\right] \\ =&\begin{cases} 0 & \text{for } n\geq 5, \\ r & \text{for } n=4. \end{cases}\end{aligned}$$

Thus we have

$$\int_{D_r}\int_0^{h(x')} u_\varepsilon^2\, dx_n dx' = \begin{cases} o(\varepsilon) & \text{for } n\geq 5, \\ O(\varepsilon) & \text{for } n=4. \end{cases} \tag{C.33}$$

By the corresponding version of ($C$.16) and ($C$.29) to ($C$.33), we get the desired result. □

**Proposition C.7.** *Let $q\in(1,p)$, where $p:=(n+2)/(n-2)$. As $\varepsilon\to 0^+$, we have*

$$\int_\Omega |w_\varepsilon|^{q+1}\, dx = \begin{cases} O(\varepsilon^{(p-q)/(p-1)}) & \textit{for } n\geq 5, \\ O\left((\varepsilon|\log\varepsilon|)^{(p-q)/(p-1)}\right) & \textit{for } n=4. \end{cases}$$

*Proof.* As $2 < q+1 < p+1$, there is a constant $\theta \in (0,1)$ such that

$$\frac{1}{q+1} = \frac{\theta}{2} + \frac{1-\theta}{p+1}. \tag{C.34}$$

Then we consider interpolation,

$$\|w_\varepsilon\|_{q+1} \le \|w_\varepsilon\|_2^\theta \cdot \|w_\varepsilon\|_{p+1}^{1-\theta}. \tag{C.35}$$

By Proposition $C.4$ and $C.6$, $(C.35)$ read as,

$$\int_\Omega |w_\varepsilon|^{q+1}\, dx = \begin{cases} O(\varepsilon^{\theta\cdot(q+1)/2}) & \text{for } n \ge 5, \\ O\left((\varepsilon|\log\varepsilon|)^{\theta\cdot(q+1)/2}\right) & \text{for } n = 4. \end{cases}$$

Finally, by $(C.34)$, we have

$$\theta = \frac{2(p-q)}{(q+1)(p-1)}, \tag{C.36}$$

and the result follows. □

**Proposition C.8.** *As $\varepsilon \to 0^+$, we have*

$$\frac{\frac{1}{2}K_1 - I(\varepsilon)}{\left[\frac{1}{2}K_4 - II(\varepsilon)\right]^{(n-2)/n}} = 2^{-2/n}S - \gamma II(\varepsilon) + o(\sqrt{\varepsilon}),$$

*for some positive constant $\gamma$.*

*Proof.* First note that $II(\varepsilon) = O(\sqrt{\varepsilon})$, we will utilize this fact several times in this proof. We thus have

$$\frac{\frac{1}{2}K_1 - I(\varepsilon)}{\left[\frac{1}{2}K_4 - II(\varepsilon)\right]^{(n-2)/n}} = \frac{\frac{1}{2}K_1 - I(\varepsilon)}{\left(\frac{1}{2}K_4\right)^{(n-2/n)} - \frac{n-2}{n}\left(\frac{1}{2}K_4\right)^{-2/n} II(\varepsilon)} + o(\sqrt{\varepsilon}). \tag{C.37}$$

To estimate the right hand side of $(C.37)$, we use integration by parts to compute as in [13] that

$$\begin{aligned} \lim_{\varepsilon\to0^+} \frac{I(\varepsilon)}{II(\varepsilon)} &= \lim_{\varepsilon\to0^+} \frac{I'(\varepsilon)}{II'(\varepsilon)} \\ &= (n-2)2\int_{\mathbb{R}^{n-1}} \frac{|y'|^2 g(y')}{(1+|y'|^2)^n}\, dy' \Big/ \int_{\mathbb{R}^{n-1}} \frac{g(y')}{(1+|y'|^2)^n}\, dy' \\ &= (n-2)^2 \int_0^\infty \frac{r^{n+2}}{(1+r^2)^n}\, dr \Big/ \int_0^\infty \frac{r^n}{(1+r^2)^n}\, dr \\ &= (n-2)^2 \frac{n+1}{n-3}, \end{aligned} \tag{C.38}$$

and that

$$\frac{n-2}{n}\frac{K_1}{K_4} = \frac{(n-2)^3}{n}\int_0^\infty \frac{r^{n+1}}{(1+r^2)^n}\,dr \Big/ \int_0^\infty \frac{r^{n-1}}{(1+r^2)^n}\,dr = (n-2)^2. \quad \text{(C.39)}$$

Combining $(C.38)$ and $(C.39)$, we get

$$\frac{I(\varepsilon)}{II(\varepsilon)} = \frac{n-2}{n}\frac{K_1}{K_4} + \gamma' + o(1),$$

which implies

$$\left(\frac{1}{2}K_4\right)^{(n-2)/n} I(\varepsilon) = \frac{n-2}{n}\left(\frac{1}{2}K_1\right)\left(\frac{1}{2}K_4\right)^{-2/n} II(\varepsilon) + \gamma'' II(\varepsilon) + o(\sqrt{\varepsilon}),$$

for some positive constant $\gamma'$, $\gamma''$, since $II(\varepsilon) = O(\varepsilon)$. We now use this estimate to compute complete the proof. Note that this ensures

$$\begin{aligned}
&\left(\frac{1}{2}K_4\right)^{(n-2)/n}\left(\frac{1}{2}K_1 - I(\varepsilon)\right) \\
&\quad = \left(\frac{1}{2}K_1\right)\left(\left(\frac{1}{2}K_4\right)^{(n-2)/n} - \frac{n-2}{n}\left(\frac{1}{2}K_4\right)^{-2/n} II(\varepsilon)\right) - \gamma'' II(\varepsilon) + o(\sqrt{\varepsilon}).
\end{aligned}$$

Thus $(C.37)$ reads as

$$\begin{aligned}
&\frac{\frac{1}{2}K_1 - I(\varepsilon)}{\left[\frac{1}{2}K_4 - II(\varepsilon)\right]^{(n-2)/n}} \\
&\quad = \frac{\frac{1}{2}K_1}{\left(\frac{1}{2}K_4\right)^{(n-2)/n}} - \frac{\gamma'' II(\varepsilon) + o(\sqrt{\varepsilon})}{\left(\frac{1}{2}K_4\right)^{(n-2)/n}\left[\left(\frac{1}{2}K_4\right)^{(n-2)/n} - \frac{n-2}{n}\left(\frac{1}{2}K_4\right)^{-2/n} II(\varepsilon)\right]} \\
&\quad = \frac{\frac{1}{2}K_1}{\left(\frac{1}{2}K_4\right)^{(n-2)/n}} - \frac{\gamma'' II(\varepsilon) + o(\sqrt{\varepsilon})}{[1 - \gamma''' II(\varepsilon)]} \\
&\quad = \frac{\frac{1}{2}K_1}{\left(\frac{1}{2}K_4\right)^{(n-2)/n}} - \gamma II(\varepsilon) + o(\sqrt{\varepsilon}),
\end{aligned}$$

for some positive constants $\gamma$, $\gamma''$ and $\gamma'''$ and we are done.

□

# Bibliography

[1] A. Ambrosetti and P. H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Functional Analysis*, 14:349–381, 1973.

[2] H. Brézis and L. Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, 36(4):437–477, 1983.

[3] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system. i. interfacial free energy. *The Journal of Chemical Physics*, 28(2):258–267, 1958.

[4] N. Dunford and J. T. Schwartz. *Linear operators. Part I.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication.

[5] N. Dunford and J. T. Schwartz. *Linear operators. Part II.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. Spectral theory. Self-adjoint operators in Hilbert space, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1963 original, A Wiley-Interscience Publication.

[6] A. El Soufi, M. Jazar, and R. Monneau. A gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann

boundary conditions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 24(1):17–39, 2007.

[7] C. M. Elliott and Z. Songmu. On the Cahn-Hilliard equation. *Arch. Rational Mech. Anal.*, 96(4):339–357, 1986.

[8] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[9] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[10] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[11] A. Novick-Cohen and L. A. Peletier. Steady states of the one-dimensional Cahn-Hilliard equation. *Proc. Roy. Soc. Edinburgh Sect. A*, 123(6):1071–1098, 1993.

[12] P. H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, volume 65 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986.

[13] X. J. Wang. Neumann problems of semilinear elliptic equations involving critical Sobolev exponents. *J. Differential Equations*, 93(2):283–310, 1991.